Durham University
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Geometry, 2023/2024

Problems Classes

Contents

1 Problems Class 1: Reflections on $\mathbb{E}^2$, geometric constructions

17 October 2023

Question 1.1. Is the following statement true of false?

"The isometries of $\mathbb{E}^2$ taking $(0,0)$ to $(0,0)$ and $(0,1)$ to $(0,2)$ form a group"

<u>Solution:</u> A map taking $(0,0)$ to $(0,0)$ and $(0,1)$ to $(0,2)$ is not an isometry. So, the set of such maps is empty. The empty set contains no identity element - which means it cannot be a group.

<u>Answer:</u> NO.

Question 1.2. Let $R_{A,\varphi}$ and $R_{B,\psi}$ be rotations with $0 < \varphi, \psi \leq \pi/2$. Find the type of the composition $f = R_{B,\psi} \circ R_{A,\varphi}$.

<u>Solution:</u> This is an example of using reflections to study compositions of isometries (we will write everything as a composition of reflections, making our choices so that some of them will cancel).

Notice that f preserve the orientation. Hence, it is either identity map, or rotation or translation. Furthermore, uniqueness part of Theorem 1.10 implies that $f = id$ if and only if $R_{A,\varphi} = R_{B,\psi}^{-1}$. In other words, $f = id$ if and only if $A = B$ and $\varphi = -\psi$.

To determine when f is a rotation and when it is a translation we write each of $R_{A,\varphi}$ and $R_{B,\psi}$ as a composition of two rotations (so that $f = r_4 \circ r_3 \circ r_2 \circ r_1$). Let l be the line through A and B. Then there exist lines l' and l'' such that $R_{B,\psi} = r_{l''} \circ r_l$ and $R_{A,\varphi} = r_l \circ r_{l'}$. Hence,

$$f = r_{l''} \circ r_l \circ r_l \circ r_{l'} == r_{l''} \circ r_{l'}.$$

Therefore f is a translation if $l'||l''$ and a rotation otherwise.

Finally, since $R_{B,\psi} = r_{l''} \circ r_l$, the angle from l to l'' is $\psi/2$. Also, since $R_{A,\varphi} = r_l \circ r_{l'}$, the angle from l' to l equals $\varphi/2$, see Fig.1. Since $0 < \varphi, \psi < \pi/2$, we see that f is always a rotation.

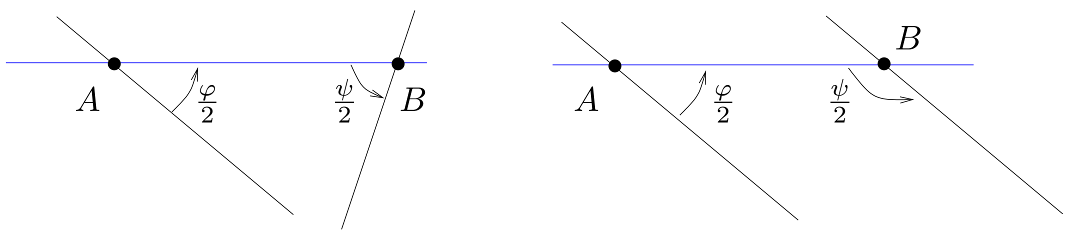

Figure 1: Question 1.1.

Question 1.3. Let A and B be two given points in one half-plane with respect to a line l. How to find a shortest path, which starts at A then travels to l and returns to B? (How to find the point where this path will reach the line l?)

Solution: Consider the point B' symmetric to B with respect to the line l. Then the shortest path from A to B' is the segment AB'. Let $M = AB' \cap l$, see Fig. 2. We claim that the broken line AMB (travelling from A to M and then to B) is the shortest path from A to B visiting a point on l.

Indeed, for any path γ from A to B visiting a point $Q \in l$ there exists a path γ' from A to Q and then from Q to B' such that the length of γ is the same as the length of γ' (we just reflect the part QB with respect to l). Since AB' is the shortest path from A to B', the broken line AMB is shorter than any other path from A to B vising the line l.

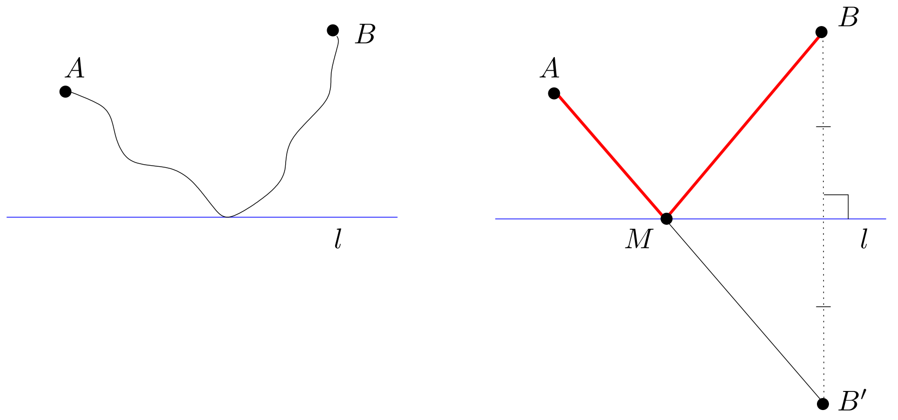

Figure 2: Question 1.2.

Question 1.4 (Geometric constructions)**.** By geometric constructions we mean constructions with ruler and compass. Here, a ruler is an instrument allowing to draw a line AB through two given points A and B. And a compass is an instrument allowing to draw a circle $C_A(AB)$ with the centre A and radius $\overline{AB}$. In this question we discuss how to construct the following sets:

(a) perpendicular bisector,

(b) midpoint of a segment,

(c) perpendicular from a point to a line,

(d) angle bisector,

(e) circumscribed circle for a triangle,

(f) inscribed circle for a triangle.

Solution:

(a) **Perpendicular bisector.** Given a segment AB, we need to construct a line l such that $l \perp AB$ and the point $M = l \cap AB$ is a midpoint for AB (i.e. $AM = MB$).

Construction: Let A and B be two points. To construct their perpendicular bisector, consider the circles $C_A(AB)$ and $C_B(AB)$ of radius AB centred at A and B respectively. Let X and Y be the two points of intersection of these two circles. (Their existence is due to continuity axiom - or we can obtain the point by a computation on $\mathbb{R}^2$). Then the line l_{XY} through the points X and Y is the perpendicular bisector for AB.

Proof: Let $M = XY \cap AB$. We need to show that $AM = MB$ and $\angle AMX = \angle BMX$. Notice that $\triangle AXY \cong \triangle BXY$ (by SSS), and hence, $\angle AXY = \angle BXY$. Furthermore, $\triangle AXM \cong \triangle BXM$ (by SAS), and hence $AM = BM$ and $\angle AMX = \angle BMX$.

Remark. Notice that we just proved that the locus of points on the same distance from A and B is the perpendicular bisector (E14).

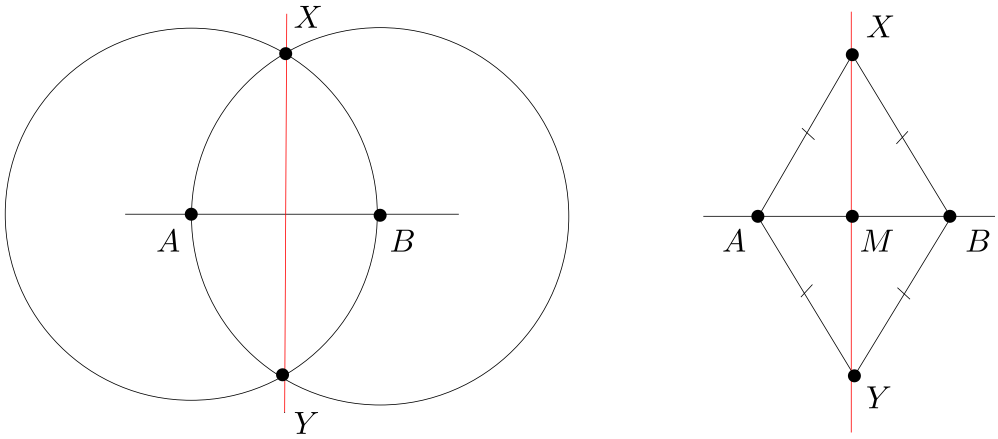

Figure 3: Question 1.3 (a): Construction of perpendicular bisector

Remark. (Extracted from the chat during the problems class).
One can do the same construction with circles centred at A and B of any equal radii - I do not need to require this radius to be AB. Then the same proof (which did not use that $AX = AB$!) will show that the construction still works. As the proof only uses that the points X, Y lie on the same (and now random!) distance from A and B, this proves in addition that the locus of points on the same distance from A and B coincides with the perpendicular bisector!

(b) **Midpoint for a segment.** This immediately follows from the construction (a).

(c) **Perpendicular from a point to a line.** Given a line l and a point $A \notin l$ we need to construct a line l' such that $A \in l'$ and $l' \perp l$.

Construction: Let $C_A(r)$ be a circle centred at A with radius $r > d(A, l)$ (where $d(A, l)$ denotes distance from A to the closest point of l). Consider the points X and Y where t $C_A(r)$ intersects l (they do exist as r is big enough). Let l' be the perpendicular bisector to XY. We claim that $A \in l'$ and $l' \perp l$.

Proof: Since $l = XY$ and l' is perpendicular to XY we have $l' \perp l$. So, we only need to prove that $A \in l'$. We know that $AX = AY$ and that the perpendicular bisector is the locus of points on the same distance from X and Y (E14), so, we conclude that $A \in l'$.

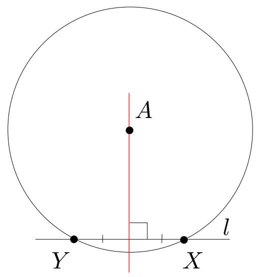

Figure 4: Question 1.3 (c): Construction of a perpendicular from a point to a line

(d) **Angle bisector.** Given an angle $\angle BAC$, we need to construct a ray AM such that $\angle BAM = \angle MAC$.

Construction: Let $C_a(r)$ be a circle centred at A of any radius r. Let $X = C_a(r) \cap AB$ and $Y = C_a(r) \cap AC$. Let l be the perpendicular bisector for the segment XY. Then l is the angle bisector for $\angle BAC$.

Proof: Notice that since $AX = AY = r$, we conclude that $A \in l$ (as the perpendicular bisector for XY is the locus of points on the same distance from X and Y by E14). Now, let $M = XY \cap l$. Then $\triangle AXM \cong \triangle YAM$ by SSS, which implies that $\angle XAM = \angle YAM$.

Remark-Exercise. An angle bisector is a locus of points on the same distance from the sides of the angle.

Hint: Given a point N on the angle bisector (resp. on the same distance from the sides of the angles), drop perpendiculars NX' and NY' on the sides of the angle and notice that $\triangle ANX' \cong \triangle ANY'$ (why?). Conclude from this that N lies on the same distance from the sides of the angle (resp. lies on the angle bisector).

(e) **Circumscribed circle for a triangle.** Given three non-collinear points A, B, C, we need to construct a circle through these points.

Construction: Let l_A be the perpendicular bisector for BC and l_B be the perpendicular bisector for AC. Then $O = l_A \cap l_B$ is the centre of the required circle.

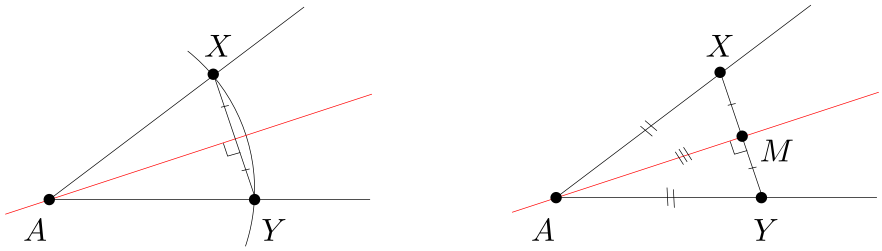

Figure 5: Question 1.3 (d): Construction of an angle bisector

Proof: We need to show that $OA = OB = OC$. Note that $OB = OC$ since $O \in l_A$, also $OA = OC$ since $O \in l_B$. This implies the statement.

Corollary. The three perpendicular bisectors in a triangle are concurrent.

Proof: As $OA = OB$, i.e. O lies on the same distance from A and B, we conclude (again by E14) that O lies on the perpendicular bisector for AB. So, the three perpendicular bisectors are concurrent at O.

(f) **Inscribed circle for a triangle.** Given a triangle ABC, we need to construct a circle which is tangent to all three sides of ABC.

Construction: Let l_A be the angle bisector for $\angle A$ and l_B be the angle bisector for $\angle B$. Then $O = l_A \cap l_B$ is the centre of the required circle. To find the radius we drop a perpendicular from O to one of the sides.

Proof: We need to show that O lies on the same distance from the lines AB, AC and BC. As $O \in l_A$, we know that O lies on the same distance from AB and AC (see the remark above!), and as $O \in l_B$, we see that O lies on the same distance from AB and CB. We conclude that O lies on the same distance from all three sides (so, if r is that distance then the circle $C_O(r)$ is tangent to all three sides and hence is the inscribed circle for $\triangle ABC$).

Corollary. The three angle bisectors in a triangle are concurrent.

Proof: As O lies on the same distance from all three sides, we conclude that it also lies on the angle bisector l_C for angle $\angle BCA$. So, three angle bisectors are concurrent at the point O.

Remarks:

- A solution for a construction question should always contain two parts:
  (i) construction (i.e. the algorithm for the construction) and
  (ii) justification (i.e. the proof that the construction provides the required object).

- One does not really need to have a ruler and a compass to solve questions on ruler and compass constructions. Moreover, I think that using the real instruments and drawing ideal diagrams does not really help to solve the questions but just distracts.

Remark on constructability. Not everything is contractible with ruler and compass. Here are several classical examples.

- **Squaring a circle:** given a circle, construct a square of the same area as the circle. This is equivalent to constructing a segment of the length $\sqrt{\pi}$ given a segment of the length 1.

- **Duplicating a cube:** given a cube of volume V construct the cube of volume $2V$. This is equivalent to constructing a segment of length $2^{1/3}$ given a segment of length 1.

- **Trisecting an angle:** Given an angle θ, construct and angle of size $\theta/3$.

For explanations why these constructions are impossible one can use field extensions, see

- Gareth Jones, *Algebra and Geometry*, Section 8.

(You will be able to find the notes by Gareth Jones on Ultra, in the Other Resources section).

2 Problems Class 2: Group actions on $\mathbb{E}^2$

1 November 2022

Question 2.1. Let $g_1, \dots, g_n$ be isometries of $\mathbb{E}^2$. Let $G = \langle g_1, \dots, g_n \rangle$ be the group generated by $g_1, \dots, g_n$ (i.e. the minimal group containing all of $g_1, \dots, g_n$). Show that the group G acts on $\mathbb{E}^2$.

Solution: By definition, $G = \{g_{i_k}^{\pm 1} \circ \dots \circ g_{i_1}^{\pm 1}\}$ (where $i_t \in \{1, 2, \dots, n\}$) is the minimal group containing $g_1, \dots, g_n$. So, this collection of the elements makes a group (as is closed, contains $e = id$, contains an inverse for every element and the operation is associative). For every $g \in G$ the element $f_g := g : \mathbb{E}^2 \to \mathbb{E}^2$ is a bijection and for every two elements $g, h \in G$ we have $f_{gh}(x) = f_g \circ f_h(x)$ for every $x \in \mathbb{E}^2$. So, G acts on $\mathbb{E}^2$.

Question 2.2. Let G be a group generated by two reflections on $\mathbb{E}^2$. When G is discrete?

Solution: Let r_1 and r_2 be reflections with respect to l_1 and l_2. Consider three cases.

- Suppose that the lines l_1 and l_2 intersect at a point O forming an angle $\alpha = \frac{p}{q}\pi$, $p, q \in \mathbb{Z}$, $q \neq 0$. Consider the set S of lines $m_1, \dots, m_q$ through the point $l_1 \cap l_2$, such that $m_1 = l_1$ and the angle between m_i and m_{i+1} equals $\frac{\pi}{q}$, see Fig. 7. Notice that applying r_1 (respectively, r_2) takes the set S to itself (not pointwise: the lines are permuted by the reflections). Furthermore, the lines $m_1, \dots, m_q$ cut the plane into $2q$ sectors, and there are only two isometries of $\mathbb{E}^2$ taking a given sector to itself. This implies that for every point $x \in \mathbb{E}^2$ the orbit $orb(x)$ of x has at most two points in any of the sectors. Hence, every orbit is finite, and hence, the group it discrete.

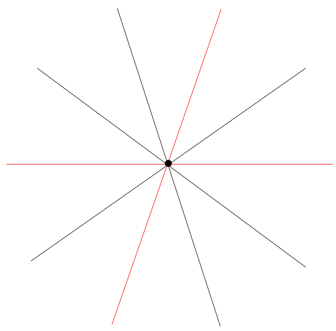

Figure 6: Question 2.1: case of rational angle (here, $\alpha = 2\pi/5$).

- Suppose that the lines l_1 and l_2 intersect at a point O forming an angle $\alpha = a\pi$ where $a \notin Q$. Then $r_2 \circ r_1$ is a rotation of infinite order. Thus, given a point $x \neq O$, the orbit $orb(x)$ contains infinitely many points on the same circle centred at O, and therefore has an accumulation point on that circle.

- Finally, suppose that $l_1 || l_2$. We leave it as an exercise to show that in this case G is always discrete.

Answer: The group is discrete unless the lines intersect forming a π-irrational angle.

Question 2.3. Let T be a triangle with angles $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}$. Let r_1, r_2, r_3 be the reflection with respect to the sides of T, and let G be the group generated by r_1, r_2, r_3. In the lecture we have checked that $G : \mathbb{E}^2$ discretely. Find a fundamental domain of this action.

<u>Solution:</u> We will show that the triangle T is a fundamental domain:

(1) It is easy to see that the images $\cup_{g\in G} g\overline{F}$ of the closure of T cover the plane.

(2) To see that $gF \cap F = \emptyset$, notice, that there are exactly 2 isometries in $Isom(E^2)$ taking F to gF for a given $g \in G$ (one of them is orientation-preserving and another is orientation-reversing). We will see that only one of these isometries lies in G.

Indeed suppose $g_1 F = g_2 F$, where g_1 is orientation-preserving and g_2 is orientation-reversing. Colour the tiling in two colours so that adjacent triangles are coloured differently and notice that each application of a generating reflection changes the colour of a triangle. In particular, this means that $g_1 F$ and $g_2 F$ should be coloured differently, and hence cannot coincide as a set.

(3) We need to check that there are finitely many elements of g such that $\overline{F} \cap g\overline{F} \neq \emptyset$. But as we have seen above, the group elements correspond to the triangles in the tiling, and every point of $\mathbb{E}^2$ belongs to at most 8 triangles. So, the statement follows.

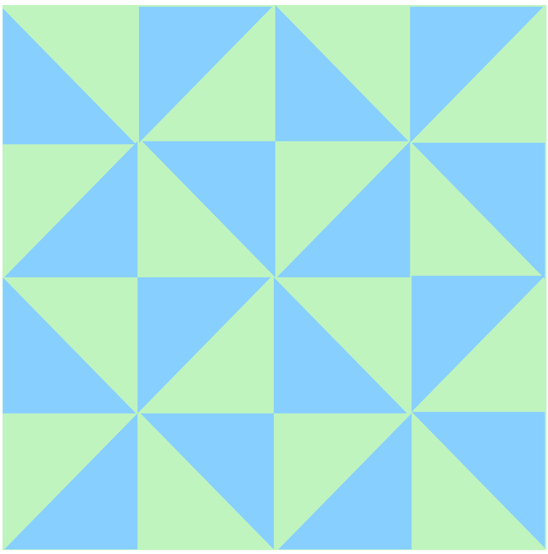

Figure 7: Question 2.2: tiling of the plane by the triangles.

Question 2.4. Find the orbit-space for the action introduced in Question 2.3.

<u>Solution:</u> We need to identify some boundary points of triangle T - when there are elements of the group G taking a point to "another" point of the boundary. But there are no such point on the boundary of the triangle. So, $T = \mathbb{E}^2/G$ is the orbit space and the distance function on $T = \mathbb{E}^2/G$ coincides with the restriction to T of the distance on $\mathbb{E}^2$.

Question 2.5. Let X be a regular triangle on $\mathbb{E}^2$. Let $G = \langle r_1, r_2 \rangle$ where r_1 and r_2 are two distinct reflections taking X to itself. Find a fundamental domain of the action $G : X$. Find also the orbit-space.

<u>Solution:</u> The triangle X can be tiles by three triangles with angles $2\pi/3, \pi/6, \pi/6$, see Fig 8. However, such a triangle is not a fundamental domain for the action. Indeed, if the three triangles are P, r_1P and r_2P then the triangles r_2P and r_1r_2PX coincide as sets with the triangle Q in the middle of Fig 8, but they does not coincide pointwise: they are mapped to each other with different orientation.

One can cut the triangle X into two halves - triangles T with angles $\pi/2, \pi/3, \pi/6$. Then it is straightforward to check that such a triangle is a fundamental domain. (The closure of it's images cover the space, interiors of the six images do not intersect, and every boundary point belongs to finitely many images). The orbit space coincides with the fundamental domain T.

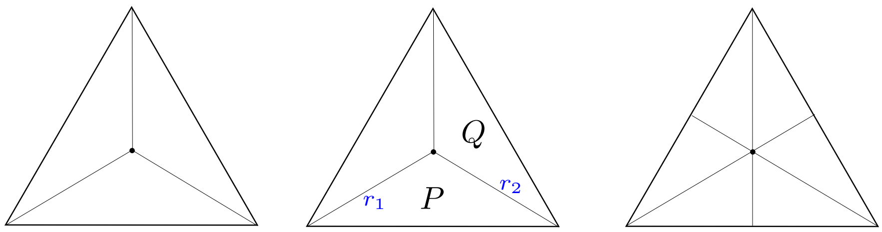

Figure 8: Question 2.4.

Question 2.6. Let G be a group generated by rotation through angle $\frac{2\pi}{3}$ on the plane. Find the orbit-space of the action $G : \mathbb{E}^2$. Are there closed geodesics in this orbit-space? Are there bi-infinite open geodesics?

<u>Solution:</u> The orbit space is a cone (which you can obtain by gluing the two boundary rays of the cone). Such a surface with a "cone singularity" - non-flat point at the tip of the cone - is called an "orbifold".

There are no closed geodesics on this orbit space, as every line cutting through a triangle with angles $2\pi/3, \pi/6, \pi/6$ will meet the two identified sides at different angles.

There are bi-infinite open geodesics (i.e. geodesics which one can extend infinitely in both directions). To see such a geodesics we just draw a line on $\mathbb{E}^2$ and take the quotient to see it's trace on the quotient space (one can see that such a geodesic will intersect itself before escaping to the infinity).

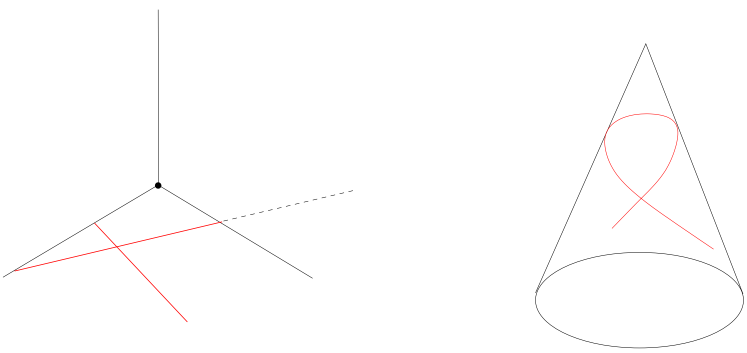

Figure 9: Question 2.5.

3 Problems Class 3: Spherical geometry

14 November 2023

Question 3.1. Let $G : S^2$ be an action. G acts discretely if and only if $|G| < \infty$.

<u>Solution:</u> If the group G is finite, then every orbit is finite and cannot have accumulation points, so the action is discrete.

Conversely, assume that $|G| = \infty$. It is enough to show that there is at least one infinite orbit, then, as S^2 is compact this orbit will have an accumulation point. To see that there is an infinite orbit, notice that an isometry of the sphere is determined by the images of 3 non-collinear points A_1, A_2, A_3. So, if the orbits $orb(A_i)$, $i = 1, 2, 3$ are all finite, then there are only finitely many possibilities for the elements of G, which contradicts to the assumption that G is an infinite group.

Question 3.2. Let $G : X$ be an action and suppose that F is its fundamental domain. Then one can show that the action $G : X$ is discrete.

<u>Idea of Solution:</u> Suppose that the action is not discrete, i.e. there are points $p, q \in X$ such that p is an accumulation point of points of orbit of q. Since F is a fundamental domain, there exist $g \in G$ such that $p \in g\overline{F}$. If $p \in gF$, then there are infinitely many points of the orbit of q in gF, and it is easy to see that there is an element $h \in G$ such that $hG \cap gG \neq \emptyset$, which implies that $G \cap h^{-1}gG \neq \emptyset$. If $p \in \partial F$, then p lies in a finite number of copies g_iF, $g \in G$, and at least one of g_iF contains infinitely many points of the orbit of q. This as before contradicts to the assumption that $g \cap gF = \emptyset$ for all $g \neq id$ in G.

Question 3.3. Let g be a reflection, $h \in Isom(S^2)$. Show that if there exists an isometry $f \in Isom(S^2)$ such that $fgf^{-1} = h$ then h is a reflection too.

<u>Solution:</u> If $h = fgf^{-1}$ for some isometry f, then $Fix_h = f(Fix_g)$ (see Proposition 1.18). Since g is a reflection, Fix_g is a line, and hence $f(Fix_g)$ is a line. We conclude that Fix_h is a line, which implies that h is a reflection by Remark 2.35.

Question 3.4. Let S^2 be a sphere of radius R. Show that the length of a circle of (spherical) radius r equals to $2\pi R \sin \frac{r}{R}$.

<u>Solution:</u> Let is find a Euclidean radius of the circle on the sphere defining the spherical circle of spherical radius r on a sphere of radius R. Let O be the centre of the sphere and N (North Pole) be the centre of the spherical circle, see Fig. 10 Then the circle is made of points X such that $\angle XON = r/R$. The spherical circle then is the intersection of the sphere with the horizontal Euclidean plane given by $z = R\cos\frac{r}{R}$, and the Euclidean radius of the circle is $R\sin\frac{r}{R}$. Hence, the length $l(C)$ of the circle is $l(C) = 2\pi R \sin\frac{r}{R}$.

Remark: We computed that for the sphere of radius R, the length of the circle of radius r will be $2\pi R \sin(\frac{r}{R})$. When $R \to \infty$ we see that $\frac{r}{R} \to 0$ and, hence, $2\pi R \sin(\frac{r}{R}) \to 2\pi r$.

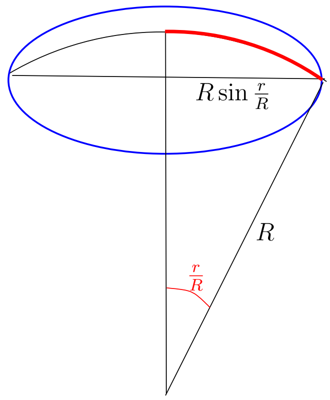

Figure 10: Question 3.4: length of a spherical circle.

Another solution: In the previous solution we used that $S^2 \subset \mathbb{E}^3$, but we can also show the same statement based on intrinsic computations (we will compute for the unit sphere here).

Consider a regular n-gon P inscribed into a circle of radius r. When $r \to \infty$, the perimeter of this n-gon tends to the length of the circle. Let A, B be two adjacent vertices of P, and let M be the midpoint of AB. Let O be the centre of the circle. Then $OA = r$, $\angle AOM = \pi/n$, and from the sine rule applied to the right-angled triangle AOM we see that

$$\frac{\sin AM}{\sin\frac{\pi}{n}} = \frac{\sin OA}{\sin\frac{\pi}{2}}$$

which implies $\sin AM = \sin r \sin(\pi/n)$. So, we obtain that the length $l(r)$ of a circle of radius r on the unit sphere is

$$l(r) = \lim_{n\to\infty} 2n \sin r \sin\frac{\pi}{n} = 2\sin r \lim_{n\to\infty} \pi\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}} = 2\pi\sin r.$$

Question 3.5. Let S^2 be a sphere of radius R. Let α and β be two parallel planes crossing S^2. Let h be the distance between α and β. Find the area of the part of S^2 lying between the planes α and γ.

Solution: Let $O = (0,0,0)$ be the centre of the sphere of radius R. We will compute the area of a very thin slice S_h of the width h defined by the planes α given by $z = z_0$ and γ given by $z = z_0 + h$. We will approximate the area of the slice with very small Euclidean rectangles whose one side approximate the circle $\mathbb{S}^2 \cap \alpha$ and the opposite side approximate $\mathbb{S}^2 \cap \beta$. Denote $AA'C'C$ the vertices of such a rectangle, where $A, A' \in \alpha$, $C, C' \in \gamma$ (here we imagine that C and C' are almost lying on the same meridians of the sphere as A and A' respectively). Then the total area of the side surface of the slice is the total length of the bases multiplied by the length AC of another side of the rectangle. Denote $B = (0,0,z_0)$ and suppose that $\angle AOB = \varphi$. Then $AB = R\sin\varphi$ is the radius of the circle $\mathbb{S}^2 \cap \alpha$, and hence, the length of that circle (the total length of all bases) equals $2\pi R\sin\varphi$. To find the length AC notice that its projection to the vertical line is of length h and the angle to the horizontal line coincides with φ (as AC is tangent to the circle, i.e. orthogonal to OA, and the horizontal line is orthogonal the vertical line). Hence, $AC = \frac{h}{\sin\varphi}$. This implies that the area S_h of a very thin h-slice can be computed as follows:

$$S_h = 2\pi BC \cdot AC = 2\pi R\sin\varphi \cdot \frac{h}{\sin\varphi} = 2\pi Rh,$$

which does not depend on α but only depends on h!

In particular, if α is given by $z = R/2$ then it cuts the upper hemisphere into two parts of equal area.

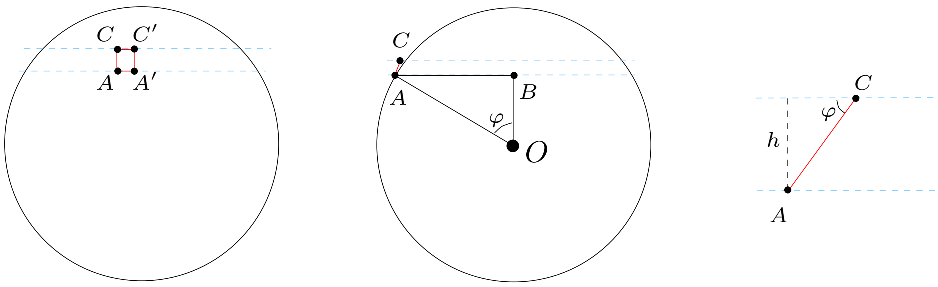

Figure 11: Question 3.5: area of a spherical slice.

Question 3.6. One can also discuss ruler and compass constructions on $\mathbb{S}^2$, similarly to the ones on $\mathbb{E}^2$ (of course, with spherical ruler and compass - which can draw spherical lines and circles).

- The following constructions will work exactly the same way as on $\mathbb{E}^2$:
  - perpendicular from a point to a line,
  - midpoint of a segment,
  - perpendicular bisector,
  - angle bisector,
  - circumscribed circle for a triangle,
  - inscribed circle for a triangle.

- Additional constructions for $\mathbb{S}^2$:
  - a pole for a line,
  - a polar line for a pole,
  - polar triangle.

- Example of a construction:
  Construct vertices of a regular tetrahedron
  (i.e. construct a triangle with angles $(2\pi/3, 2\pi/3, 2\pi/3)$*).*

  One can show that the following steps give the required construction:
  - Draw any regular triangle;
  - Construct angle $2\pi/3$;
  - Construct length $2\pi/3$ (by crossing the sides of angle $2\pi/3$ with the line polar to the vertex of the angle);
  - Construct length $\pi/3$ (by taking a midpoint);
  - Construct a triangle with the lengths $(a, b, c) = (\pi/3, \pi/3, \pi/3)$;
  - By Bipolar Theorem, the polar triangle has required angles $(2\pi/3, 2\pi/3, 2\pi/3)$).

- Can you construct the vertices of an octahedron and a cube?

- Given an angle $\pi/5$, can you construct the vertices of an icosahedron and a dodecahedron?

Here you can find a construction of vertices of iscosahedron and dodecahedron (and an instruction how to easily draw both on paper).

4 Problems Class 4: Projective geometry

28 November 2023

Question 4.1. Find a projective transformation f which takes

$$\begin{aligned} A &= (1:0:0) \text{ to } (0:0:1) \\ B &= (0:1:0) \text{ to } (0:1:1) \\ C &= (0:0:1) \text{ to } (1:0:1) \\ D &= (1:1:1) \text{ to } (1:1:1) \end{aligned}$$

Find the image of $X = AD \cap BC$ under this transformation f.

<u>*Solution:*</u> We will search for this transformation as for a 3×3 matrix with indefinite coefficients. First, from looking at the image of the point A we know

$$\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = k \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

This implies $a = b = 0$, and we may assume $c = 1$ ($c \neq 0$ as $\det A \neq 0$). Next, using the point B we know that

$$\begin{pmatrix} 0 & d & g \\ 0 & e & h \\ 1 & f & i \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} d \\ e \\ f \end{pmatrix} = l \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

hence, $d = 0$ and $e = f$. Furthermore, from the point C we have

$$\begin{pmatrix} 0 & 0 & g \\ 0 & e & h \\ 1 & e & i \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} g \\ h \\ i \end{pmatrix} = m \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

hence, $h = 0$ and $i = g$. Finally, from the point D we get

$$\begin{pmatrix} 0 & 0 & g \\ 0 & e & 0 \\ 1 & e & g \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} g \\ e \\ 1 + e + g \end{pmatrix} = n \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

which implies $g = e = 1 + e + g$, i.e. $e = 1 + 2e$, and hence $e = -1$. So, we arrive to the projective transformation given by

$$f = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & -1 & -1 \end{pmatrix}.$$

To find the image of $X = AD \cap BC$, consider the points A, B, C, D in the unit cube with vertex $O = (0, 0, 0)$, see Fig. 12. Then BCO span the plane $x = 0$ and AOD span the plane $y = z$. These two planes intersect by the line through the point $X = (0, 1, 1)$. The image $f(X)$ of this point is

$$f(X) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix}.$$

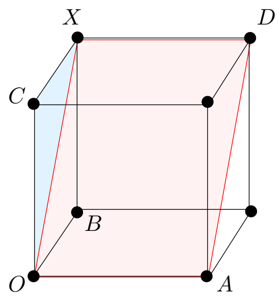

Figure 12: Question 4.1.

Question 4.2. Find $[A, B, C, D]$ for the points above. (Does it exist?)
For $E = (1 : 1 : 0)$, $F = (1 : 2 : 0)$ find $[A, B, E, F]$.

Solution: The cross-ratio $[A, B, C, D]$ is not defined as the points do not lie on one line.

The points A, B, E, F all belong to the line defined by $z = 0$, so we can find their cross-ratio using cross-ratio of the four lines OA, OB, OE, OF. This cross-ratio can be computed as a cross-ratio of the points obtained by intersection of the lines with any given line l. Choose l to be the line $x = 1$ (on the plane $z = 0$), see Fig. 13, left. Notice that the points A, E, F already lie on that line, and the intersection of OB with the line l is $B' = \infty$ Then

$$[A, B, E, F] = \frac{|EA|}{|EB'|} \Big/ \frac{|FA|}{|FB|} = \frac{1}{\infty} \Big/ \frac{2}{\infty} = \frac{1}{2}$$

Remark: If you don't trust this computation due to $B' = \infty$, you can cross the four lines by any other line lying in the plane and check that you get the same answer. (For example, if you chose the line $2x + y = 2$ the computation is still very short and the numbers are nice).

Question 4.3. Check explicitly, that the transformation f from Question 1 preserves the value of $[A, B, E, F]$.

Solution: We know the images $f(A)$ and $f(B)$. Let us compute $f(E)$ and $f(F)$:

$$f(E) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \qquad f(F) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}.$$

The points all four points $f(A), f(E), f(F)$ and $f(B)$ lie in the plane $x = 0$ (see Fig. 13, right), so, the corresponding points of the projective plane collinear

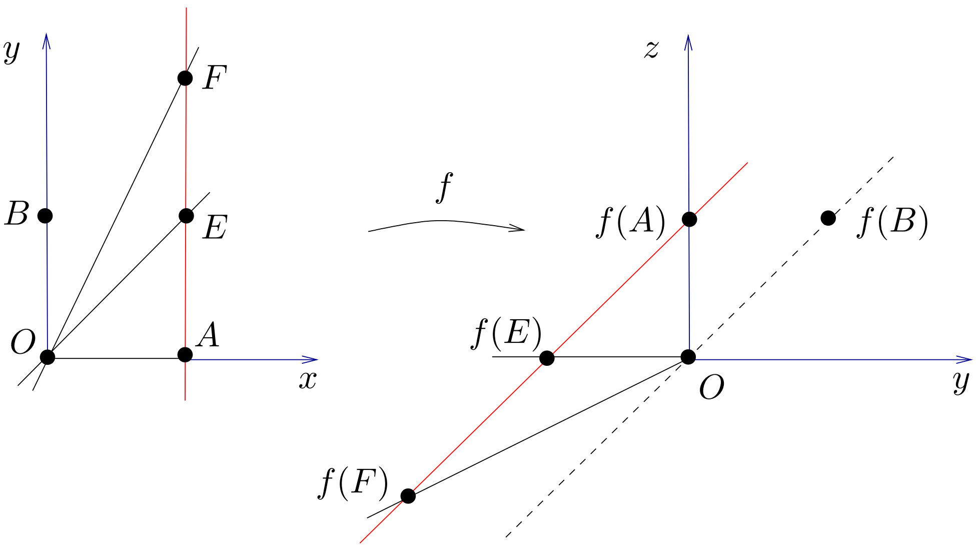

Figure 13: Questions 4.2 and 4.3.

and the cross-ratio makes sense. Furthermore, inside the plane $x = 0$, the points $f(A), f(E), f(F)$ lie in the line $y - z = -1$ (of the plane $x = 0$), so, we project $f(B)$ to the point B'' of the same line. Then we get

$$[f(A), f(B), f(E), f(F)] = [0, \infty, 1, 2] = \frac{1-0}{1-\infty} \Big/ \frac{2-0}{2-\infty} = \frac{1}{2},$$

which agrees with the computation for $[A, B, E, F]$ above.

Question 4.4. Let A_1, A_2, A_3, A_4 be points on a line a, let B_1, B_2, B_3, B_4 be points on a line b. Denote by p_i the line through A_i and B_i. Show that if the lines p_1, p_2, p_3, p_4 are concurrent, then the points $A_{i+1}B_i \cap A_iB_{i+1}$ $(i = 1, 2, 3)$ are collinear.

<u>Solution:</u> To show that the points are collinear, we apply a projective transformation (which preserves collinearity), so that the configuration will get simpler. Namely, let P be a point where the lines p_i meet, and let $Q = a \cap b$. Consider a projective transformation f which takes the line PQ to the line at infinity. Then f takes the lines $p_1, \dots, p_4$ to four parallel lines and the lines a and b to a pair of parallel lines, see Fig. 14. Applying an affine transformation g we can assume that the two lines obtained from a, b are orthogonal to the four line obtained from $p_1, \dots, p_4$. The configuration we obtained consists of 3 rectangles attached to each other back to back - and the points considered in the question are mapped to the centres of these three rectangles, which obviously lie in one line (parallel to $g(f(a)$ and $g(f(b)))$. Hence, the original points are also collinear.

Question 4.5. Formulate and prove the statement dual to the one in Question 4.

<u>Solution:</u> Here is the dual statement:

Given the points A and B and the lines a_1, a_2, a_3, a_4 though A and lines b_1, b_2, b_3, b_4 through B, consider the points $P_i = a_i \cap b_i$. Let $Q_i = a_{i+1} \cap b_i$ and $R_i = a_i \cap b_{i+1}$ for $i = 1, 2, 3$. If the points $P_1, \dots, P_4$ are collinear, then the lines Q_iR_i are concurrent.

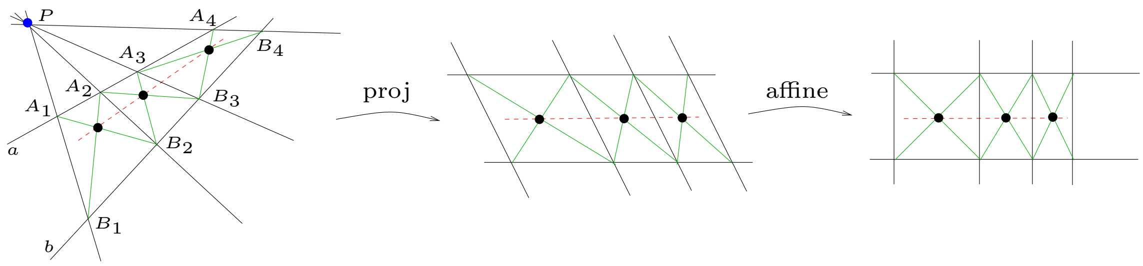

Figure 14: Question 4.4.

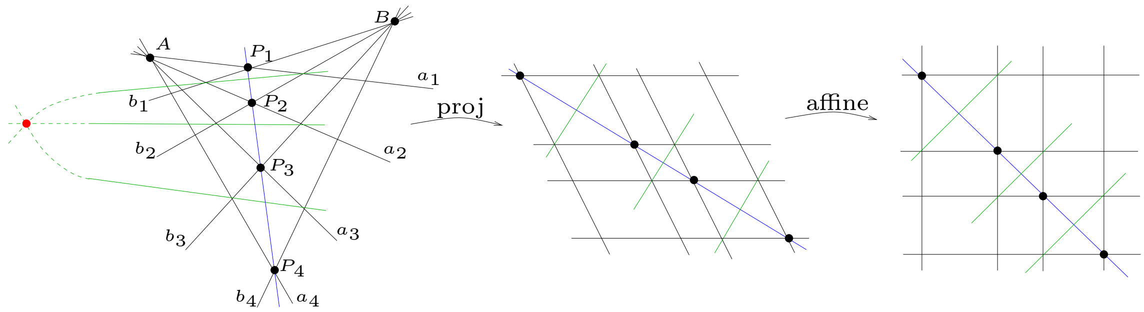

Figure 15: Question 4.5.

To prove the statement, we map (by a projective map f) the points A and B to points on the line at infinity. Applying additionally an affine map g , we may assume that we obtain a configuration of four parallel lines intersected orthogonally by another four parallel lines. The configuration looks like a table of 3×3 rectangles with the points $g(f(P_i))$ lying "on the main diagonal". Moreover, we can assume that the images of the points P_1 and P_2 lie on the diagonal of a square, not a general rectangle. Then from the assumption that the points P_1, P_2, P_3, P_4 are collinear, we see that the points P_2 and P_3 are also lying on a diagonal of a square, and the same is true for P_3 and P_4. Then, all three lines Q_iR_i are also diagonal of the squares - so they are parallel to each other (i.e. concurrent at some point at infinity).

Remark. (On using transformation groups to simplify questions).
Here is what we really did in solutions of Questions 4.4 and and 4.5 above:

1. We get a question about points and lines in $\mathbb{R}^2$.

2. We notice that the question only deals with properties preserved by projective map.

3. Hence, we consider $\mathbb{R}^2$ as a (finite) part of $\mathbb{RP}^2$ (embedded to $\mathbb{RP}^2$ as intersection of all objects with the plane $z = 1$).

4. Apply a projective map to simply the questions.

5. Solve the simplified question.

6. Since the projective transformation preserves the properties we are looking at, we can conclude about the original, more harder, question.

5 Problems Class 5: Möbius Geometry

24 January 2023

Question 5.1. Find the type of Möbius transformation $f(z) = 1/z$.

Solution: A Möbius transformation is parabolic if it has exactly one fixed point and non-parabolic otherwise. Since the equation $z = \frac{1}{z}$ is equivalent to $z^2 = 1$, which has two roots $z = \pm 1$, we conclude that f is not parabolic.

A non-parabolic Möbius transformation can be elliptic, hyperbolic or loxodromic. Elliptic transformations have two fixed points with similar properties while hyperbolic and loxodromic transformations have one repeller and one attractor. Furthermore, $f(f(z)) = \frac{1}{1/z} = z$ is the identity map, i.e. the orbit of a point $w \in \mathbb{C}$, $w \neq \pm 1$ contains exactly two points (w and $1/w$), which means that f has no repeller or attractor, i.e. f is an elliptic transformation (of order 2).

Answer: f is elliptic.

Question 5.2. Let f and g be inversions with respect to two intersecting circles. Show that $g \circ f = f \circ g$ if and only if $Fix_f \perp Fix_g$.

Solution: Let $A \in Fix_f \cap Fix_g$ be an intersection point, and let h be a Möbius transformation sending A to ∞. As Möbius transformations preserve angles, $Fix_f \perp Fix_g$ if and only if $h(Fix_f) \perp h(Fix_g)$. Also, $h(Fix_f) = Fix_{hfh^{-1}}$ and $h(Fix_g) = Fix_{hgh^{-1}}$; and furthermore, f commutes with g if and only if hfh^{-1} commutes with hgh^{-1}.

So, the statement in the question is equivalent to the following:

$$hfh^{-1} \text{ commutes with } hgh^{-1} \text{ if and only if } h(Fix_f) \perp h(Fix_g).$$

Since both $h(Fix_f)$ and $h(Fix_g)$ contain the point ∞, these sets are lines, and hence the transformations hfh^{-1} and hgh^{-1} are reflections. A composition of two reflections is a translation (if the lines are parallel) or a rotation (if the lines are not parallel), and it is easy to check that they commute if and only if the lines intersect each other orthogonally.

Question 5.3. Let $\mathcal{C}_1, \ldots, \mathcal{C}_5$ be circles all passing through the points A and B on the plane. Show that there exists a circle γ orthogonal to all of $\mathcal{C}_i$.

Solution: Let f be a Möbius transformation taking A to infinity. Then f takes the circles $\mathcal{C}_i$ to lines through $f(B)$. Any circle γ centred at B is orthogonal to all of $\mathcal{C}_i$. Then its preimage $f^{-1}(\gamma)$ is a line or circle orthogonal to all of $\mathcal{C}_i$ (indeed, since f is a Möbius transformation, f^{-1} is a Möbius transformation too, and hence it takes lines to circles and lines and preserves angles). Notice that we have infinitely many choices of γ, and only one of them can pass through the point $f(\infty)$, which implies that only for that one choice of γ the preimage $f^{-1}(\gamma)$ is a lines instead of a circle, while every other choice provides us with a required circle.

Question 5.4. Prove Ptolemy's theorem: given a quadrilateral $ABCD$ inscribed into a circle one has

$$|AC| \cdot |BD| = |AB| \cdot |CD| + |BC| \cdot |AD|.$$

<u>*Solution:*</u> We will mention 3 proofs of this remarkable theorem.

(a) (Cross-ratio) This is the question 12.1 from the Assignment 11-12, and the Hints suggest a scheme for a solution based on cross-ratio.

(b) (Inversion). Let $\mathcal{C}$ be the circle containing the points A, B, C, D. Let γ be a unit circle centred at D and let I_γ be inversion with respect to γ. Then $I_\gamma(D) = \infty$ and I_γ takes $\mathcal{C}$ to a line l. Let $A', B', C' \in l$ be the images of A, B, C, see Fig. 16).

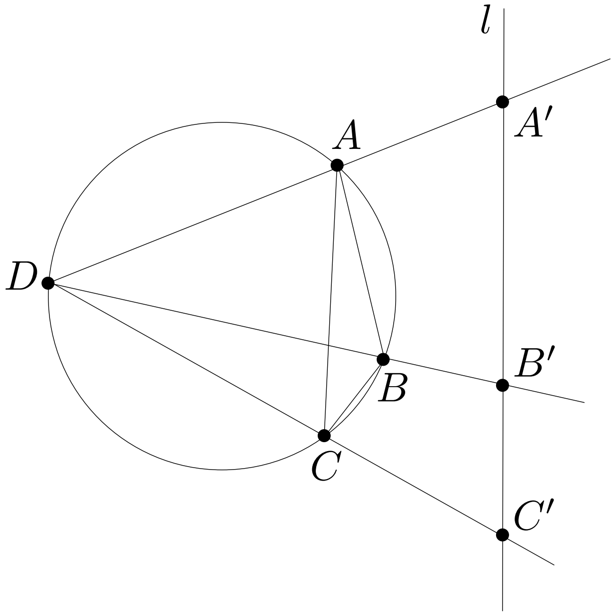

Figure 16: Ptolemy theorem via inversion.

From three pairs of similar triangles we have

$$\frac{|AC|}{|A'C'|}=\frac{|AD|}{|C'D|}=\frac{|CD|}{|A'D|}; \qquad \frac{|AB|}{|A'B'|}=\frac{|BD|}{|A'D|}=\frac{|AD|}{|B'D|}; \qquad \frac{|BC|}{|B'C'|}=\frac{|BD|}{|C'D|}=\frac{|CD|}{|B'D|}. \tag{5.1}$$

As A', B', C' are collinear, we have $|A'C'| = |A'B'| + |B'C'|$. Expressing these values from (5.1) we obtain

$$\frac{|AC| \cdot |C'D|}{|AD|} = \frac{|AB| \cdot |A'D|}{|BD|} + \frac{|BC| \cdot |C'D|}{|BD|}.$$

Now, we would be able to cancel $C'D$ from everywhere except for the first summand, where we have $A'D$ instead. So, we will use (5.1) again to rewrite $|A'D| = \frac{|CD| \cdot |C'D|}{|AD|}$ and then cancel $|C'D|$:

$$\frac{|AC|}{|AD|} = \frac{|AB| \cdot |CD|}{|BD| \cdot |AD|} + \frac{|BC|}{|BD|}.$$

Multiplying finally by $|BD| \cdot |AD|$ we get the Ptolemy theorem.

Remark. This proof shows that the inversion turns Ptolemey's equation to degeneration of triangle inequality. Taking a point B not in the circle through

A, C, D (i.e. taking four points not on a circle) we could do the same and obtain a points B' not lying on the segment $A'C'$, and hence satisfying triangle inequality $|A'C'| < |A'B'| + |B'C'|$. Then the same computation as above will show that $|AC| \cdot |BD| < |AB| \cdot |CD| + |BC| \cdot |AD|$.

This proves that $|AC| \cdot |BD| \leq |AB| \cdot |CD| + |BC| \cdot |AD|$ for any four points on the plane, with equality taking place if and only if the points lie on one line or circle.

(c) (Proof without words). See Fig. 17 (here $x \cdot \triangle PQR$ means that we take the triangle PQR and scale it by the length x).

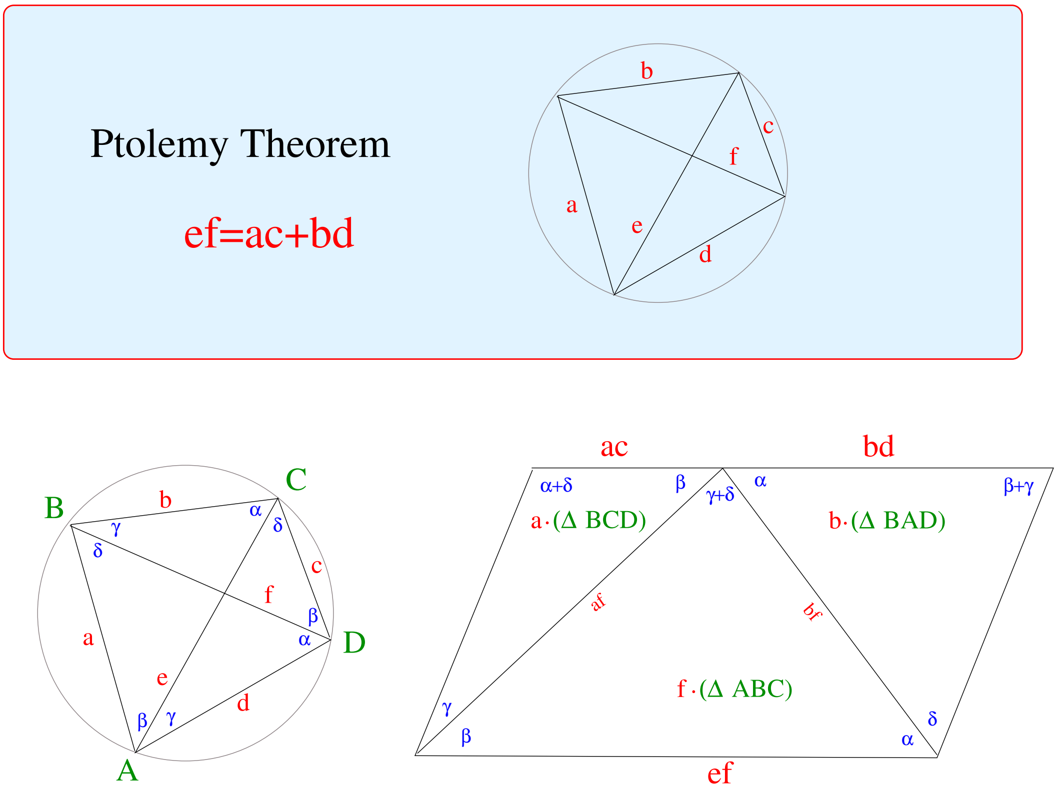

Figure 17: Ptolemy theorem: proof without words.

Remark: I have borrowed the "Proof without words" for Ptolemy theorem from the cut-the-knot portal. Which in its turn refers to the following paper:

- W. Derrick, J. Herstein, *Proof Without Words: Ptolemy's Theorem*, The College Mathematics Journal, v. 43, n. 5, November 2012, p 386.

Remark: Ptolemy's theorem implies Pythagorean: to see this assume that $ABCD$ is a rectangle and write down the statement Ptolemy Theorem.

6 Problems Class 6: Poincaré disc

6 February 2024

Question 6.1. Let l_1 and l_2 be two divergent lines. Show that there exists a unique common perpendicular to l_1 and l_2.

Solution 1: Take the line l_1 into a diameter of the disc by an isometry (we can do it by transitivity of isometries on the points of $\mathbb{H}^2$. Let $A(t)$, $t \in (-\infty, \infty)$ be a point running along l_1 from one endpoint to another. Let $m(t)$ be a line through $A(t)$ orthogonal to l_1. Then $m(t)$ first does not intersect l_2, then intersects it at a very acute angle, then at an angle growing monotonically from 0 to π and then does not intersect it again. As the angle changes continuously, it is clear that there is an intermediate position when $m(t)$ is orthogonal to l_2. As the angle grows monotonically, clearly such a line is unique.

Solution 2: Let A_1, B_1 be the endpoints of l_1 and A_2, B_2 be the endpoints of l_2 (where A_1, B_1, B_2, A_2 is the cyclic order of the points along the absolute). Let $X = A_1B_2 \cap B_1A_2$. Map the point X to the centre O of the disc. Then we obtain a very symmetric picture (with points A_1, A_2, B_2, B_1 at vertices of a rectangle). It is easy to see that in this picture the diameter (parallel to one side of the rectangle) is orthogonal to both l_1 and l_2.

Uniqueness: Consider a reflection with respect to a common perpendicular. It swaps A_1 with B_1 and A_2 with B_2. Notice that as an isometry is uniquely determined by images of three points of the absolute, there is a unique isometry swapping these points. So, there exists a unique line orthogonal to l_1 and l_2.

Question 6.2. Let $0 \leq \alpha, \beta, \gamma < \pi$, $\alpha + \beta + \gamma < \pi$. Then there exists a hyperbolic triangle with angles α, β, γ.

Solution: Consider the angle of size α at the centre of the disc (with sides of the disc being halves of diameters l_1 and l_2). Let $B(t)$, $t \in (0, \infty)$ be a point running along l_1 from the centre to the absolute. For each point $B(t)$ we can consider a ray $l_3(t)$ (a half of a hyperbolic line) starting from $B(t)$ and making the angle β with l_1. Then, when $B(t)$ moves from the centre of the disc to the absolute, the angle between $l_3(t)$ and l_2 decreases (monotonically) from $\pi - \alpha - \beta$ to 0 (as a hyperbolic line passing almost through the origin is almost straight in Euclidean sense). As the angle changes continuously, there is a position of $B(t)$ when it is equal to γ.

Question 6.3. Show that every triangle in $\mathbb{H}^2$ has an inscribed circle.

Solution: Consider an angle bisector (for an angle with vertex at the centre of the disc it is represented by Euclidean angle bisector). By symmetry, the angle bisector is the locus of points on the same distance from two sides of the angle. From this, we conclude that a point of intersection of two angle bisectors in a hyperbolic triangle lies on equal distances from all three sides (and that the three angle bisectors intersect at this point). In particular, there exists an inscribed circle.

Question 6.4. Given a hyperbolic triangle, is it true that is always has a circumscribed circle?

Solution: As in Euclidean case, it is easy to show that a perpendicular bisector is the locus of points on the same distance from two given points (to do it one could use congruence of triangles, or just symmetry - especially convenient after mapping the midpoint of the segment to the centre of the disc).

It is tempting to conclude from this that the point of intersection of two perpendicular bisectors is the centre of the circumscribed circle. And it is - when such a point does exists. At the same time, one can have triangles, where perpendicular bisectors to the sides do not intersect (draw a triangle with one very long side occupying almost whole of one of the diameters in the disc and the third vertex close to the centre of the disc). Such a triangle has no circumscribed circle (as no point of $\mathbb{H}^2$ lies on the same distance from all three its vertices).

Another attempt: Given a hyperbolic triangle ABC, there exists a Euclidean circle through the points A, B, C. Euclidean circles are hyperbolic circles in the Poincaré disc... - but only when they belong to the inside of the model! (And for the example in the paragraph above it will not...).

Answer: No.

Question 6.5. Ruler and compass constructions in hyperbolic plane, in particular:

- midpoint of a segment;
- perpendicular bisector;
- angle bisector;
- centre of a given circle;
- tangent line to a given circle at a given point of the circle;
- centre of inscribed circle for a triangle;
- centre of circumscribed circle for a triangle (when exists);
- common perpendicular for two divergent lines (in assumptions that one is given an infinitely long ruler, which allows to draw lines through two points of the absolute).

Solution: All the constructions (except for the last one) are the same as in Euclidean or spherical case.

For the last one we connect the pairs of endpoints of two lines A_1B_1 and A_2B_2 and find the intersection $X = A_1B_2 \cap A_2B_1$ of the obtained lines. Then we drop a perpendicular l' from X to A_1B_1 and from the symmetry (as in the first question) we see that l' is also orthogonal to A_2B_2.

Remark. I don't know whether it is possible to construct with hyperbolic (finite) ruler and compass the following hyperbolic objects:

- a common perpendicular to two lines;
- a tangent line to a circle passing through a given point outside of the circle; (maybe I have seen this, but I cannot remember where...)
- Simultaneous tangent line to two given circles.

Question 6.6. A polygon with all vertices at the absolute is called *ideal*. Is it true that all ideal hyperbolic quadrilaterals are isometric to each other?

<u>*Solution:*</u> First, notice that all ideal triangles are indeed congruent to each other (by triple transitivity of isometries on points of the absolute).

Given ideal quadrilaterals $ABCD$ and $A'B'C'D'$, we can always map A, B, C, to $A'B'C'$ by an isometry f (by triple transitivity). However, such an isometry is unique. So, if $f(D) \neq D'$, then the quadrilaterals $ABCD$ and $A'B'C'D'$ are not isometric.

Remark. In fact, there is a 1-parametric family of quadrilaterals: when A, B, C are fixed, the point D can move along the boundary $\partial\mathbb{H}^2$. The geometric meaning of this parameter can be best seen as follows. Drop a perpendicular BH_B from B to the line AC, and similarly, drop a perpendicular DH_D from D to the line AC. In the very symmetric case (for example, when $A = -1,\ B = -i,\ C = 1,\ D = 1$) the points H_B and H_D (the feet of the perpendiculars) will coincide (and coincide with the origin O). However, when the point D will start moving from i to 1 along the boundary of the disc, H_D will move from O to 1 along the diameter AC, while H_B will stay put at O. So the distance $d(H_B, H_d)$ will grow from 0 to ∞.

7 Problems Class 7: Some computations...

20 February 2024

Question 7.1. Show that the area of a hyperbolic disc of radius r is $4\pi\sinh^2(\frac{r}{2})$.

<u>*Solution:*</u> We will compute the area S_r of the disc of radius r as a limit of area of a regular n-gons having a circumscribed circle of radius r:

$$S_r = \lim_{n\to\infty} S_{n-\text{gon}}.$$

We decompose the regular n-gon into $2n$ right-angled triangles with angles $(\frac{\pi}{2}, \frac{\pi}{n}, \alpha)$ opposite to the sides of length r, b, a respectively, see Fig. 18. To find the area of the right-angled triangle we need to find the angle α. We will compute $\cos\alpha$ instead:

$$\cos^2\alpha \overset{(HW15.2a)}{=} \frac{\tanh^2 b}{\tanh^2 r} = \frac{1}{\tanh^2 r}\frac{\sinh^2 b}{\cosh^2 b} = \frac{1}{\tanh^2 r}\frac{\sinh^2 b}{1+\sinh^2 b}$$
$$\overset{(HW15.2b)}{=} \frac{\cosh^2 r}{\sinh^2 r}\cdot\frac{\sinh^2 r\sinh^2\frac{\pi}{n}}{1+\sinh^2 r\sinh^2\frac{\pi}{n}}. \tag{7.1}$$

From this we conclude that

$$\cos\alpha = \frac{\cosh r\sin\frac{\pi}{n}}{\sqrt{1+\sinh^2 r\sin^2\frac{\pi}{n}}} = \sin(\frac{\pi}{2}-\alpha).$$

Taking in account that when $n\to\infty$ we have $\alpha\to\pi/2$, i.e. $\sin(\frac{\pi}{2}-\alpha)\to\frac{\pi}{2}-\alpha$, we conclude that

$$\alpha \sim \frac{\cosh r\sin\frac{\pi}{n}}{\sqrt{1+\sinh^2 r\sin^2\frac{\pi}{n}}} \sim \frac{\pi}{2} - \frac{\cosh r\cdot\frac{\pi}{n}}{1}.$$

Finally, we use the angle α to find the area S_r of the disc of radius r:

$$S_r = \lim_{n\to\infty} S_{n-\text{gon}} = \lim_{n\to\infty}((n-2)\pi - 2n\alpha) = \lim_{n\to\infty}((n-2)\pi - 2n(\frac{\pi}{2} - \cosh r\frac{\pi}{n}))$$
$$= -2\pi + 2\pi\cosh r = 2\pi(\cosh r - 1) = 4\pi\sinh^2(\frac{r}{2}),$$

where the last equality is shown in HW 15.3.

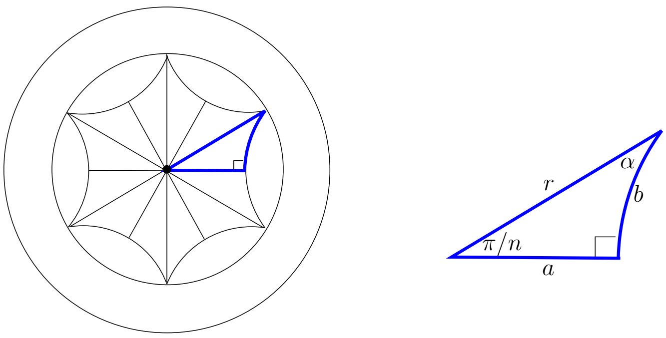

Figure 18: Computing area of hyperbolic disc.

Remark. Strictly speaking, the computation above only shows that the area of a disc is greater or equal to $4\pi\sinh^2(\frac{r}{2})$. It is easy to show that it is also smaller or equal than the same value by considering regular polygons with inscribed circle of radius r.

Question 7.2 (Hyperbolic oranges)**.** Consider hyperbolic oranges, i.e. a discs of radius r, where the inner disc of radius $\frac{9}{10}r$ is a pulp, while the outer $1/10$ is a peal. Assuming that all oranges are equally tasty, which oranges are better to buy: big or small?

<u>*Solution:*</u> Let us check how large is the part of the peel in the orange, especially when the radius r is large (we will denote by S_{orange} and S_{pulp} the corresponding areas):

$$\frac{S_{\text{orange}}}{S_{\text{pulp}}} = \frac{4\pi\sinh^2\frac{r}{2}}{4\pi\sinh^2(\frac{9}{10}\frac{r}{2})} = \left(\frac{e^{r/2}-e^{-r/2}}{e^{\frac{9}{10}\frac{r}{2}}-e^{-\frac{9}{10}\frac{r}{2}}}\right)^2 \overset{r\to\infty}{\sim} e^{r(1-\frac{9}{10})} = e^{\frac{9}{10}} \to \infty,$$

which means that when the radius of the orange grows, most part of the orange (i.e. as large part as we want) is the peel.

Question 7.3 (Spherical oranges)**.** Assuming that the spherical oranges have the same part as peel (i.e. $\frac{9}{10}r$ is a pulp, while the outer $1/10$ is a peal), how much can you eat from the best spherical orange?

Question 7.4 (Playing golf)**.**
Where is it better to play golf: on Euclidean field or on hyperbolic?

<u>*Solution:*</u>

- **Euclidean golf.** Suppose your ball (denote it B) is in $300m$ from the goal (denote it G), and you send the ball to the distance $300m$ with just a 1^o mistake. How far from the goal G is the place B' where you result?

  We can first estimate it by the $1/360$ part of the circle of radius $R = 300m$ around B, it gives:
  $$l = \frac{2\pi R}{360} = \frac{2\pi\cdot 300}{360} = 5.235987756m.$$

  Now, we really compute the distance using Euclidean cosine rule $c^2 = a^2 + b^2 - 2ab\cos\gamma$:

  $$d_E(B', G) = \sqrt{a^2 + a^2 - 2a^2\cos\frac{\pi}{180}} = 300\cdot\sqrt{2}\sqrt{1-\cos\frac{\pi}{180}} = 5.235921299m.$$

  (Notice, how precise was our first estimate!)

- **Hyperbolic golf.** Now, consider the same situation in hyperbolic settings. The length of circle estimate will give

  $$l = \frac{2\pi\sinh R}{360} = \frac{2\pi\sinh(300)}{360} = 1.695\times 10^{128}m.$$

  This inhumanly large number cannot be a good estimate, as by triangle inequality, the distance $d_H(B', G)$ between your ball and the goal is not larger than $d_H(B, B') + d_h(B, G) = 300m + 300m = 600m$!

  So, we need to compute the distance with hyperbolic cosine rule $\cosh c = \cosh a\cosh b - \sinh a\sinh b\cos\gamma$:

  $$\cosh d(B', G) = \cosh^2(300) - \sinh^2(300)\cdot\cos\frac{\pi}{180} = 1.436623167\times 10^{256}.$$

Given $x = \cosh d(B', G)$, we can find d as function of x by

$$d = \ln(x + \sqrt{x^2 - 1} \sim 590.517226324m.$$

We see that we got a good approximation again - but this time by going from B' back to initial position B and then to the goal G!

Remark. The above question is borrowed from the following video:
Playing Sports in Hyperbolic Space - Dick Canary in Numberphile (video by B. Haran).

Question 7.5. Which statements about hyperbolic triangles are applicable to unbounded triangles? (What about congruence theorems? Formulae?)

Discussion:
1. AAA congruence. First, we consider AAA congruence rules for three types of unbounded triangles (with 3,2,1 vertices at the absolute respectively). Notice, that a vertex at the absolute corresponds to a zero angle at that vertex (and no other vertices have zero angles). So, we need to compare only the triangles with the same number of vertices at the absolute.

- 3 vertices at the absolute. All such triangles (ideal triangles) are congruent in view of triple transitivity of isometries on the points of absolute.
- 2 vertices at the absolute. We can map the third vertex to the centre of Poincaré disc and fix two rays forming a given angle α (in a unique way up to isometry). Then the triangle is completely defined. So, AAA also holds in this case.
- 1 vertex at the absolute. Let the angles be $0, \alpha, \beta$. We place the vertex with angle β to the centre of the disc O, and let $X \in \partial H^2$ be the vertex at the absolute. Let OY be the ray such that $\angle XOY = \alpha$. We need to show that the ray OY contains at most one (actually, a unique) point A such that $\angle OAX = \alpha$. This is indeed the case, as if there where two such points A and A', then the sum of angles of triangle XAA' would be π, which is impossible.

So, we conclude that AAA holds even for unbounded triangles.

2. SSS congruence. A triangle with one vertex at the absolute and a finite side of length a can be right-angled or not right-angled, but in both cases the lengths of sides are (a, ∞, ∞). Similarly, a triangle with two vertices at the absolute may have any given angle at the third vertex, but the lengths of sides in this case are all infinite.

We conclude, that SSS is only applicable for the finite values of length.

Exercise: find similar examples demonstrating that SAS and ASA are also only applicable when all "S" entries correspond to finite sides.

Next, we consider two examples of using **formulae** for infinitely large triangles:

- Consider a right-angled triangle with one vertex at the absolute. Let a be the length of the finite side and φ be the size of the non-zero and non-right angle. Trying to apply sine rule to this triangle we obtain:

$$\frac{\sinh a}{\sin 0} = \frac{\sinh \infty}{\sin \varphi},$$

i.e. we get $\infty = \infty$, which is correct, but useless.

- Consider the second cosine law:

$$\cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma\cosh a.$$

  Applying it to a (bounded) right-angled triangle with $\gamma = \pi/2$ and $\beta = \varphi$, we get

$$\cos\alpha = \sin\varphi\cosh a.$$

  Now, suppose that we fixed the directions of both sides meeting at the right angle, and we keep the length of one side equal to a as it was, but increase the length of the other side to infinity. Then the angle α will approach 0, so that $\cos\alpha \to 1$, and we obtain

$$1 = \sin\varphi\cosh a,$$

  which recovers the formula for angle of parallelism $\cosh a = \frac{1}{\sin\varphi}$. Which makes sense and is very useful!

Conclusion: Formulae should work when make sense (i.e. both sides are defined and finite), but one needs to be careful. Congruence rules also work when you are comparing finite things.

It is also easy to see that the formula of area still holds for unbounded triangles (recall that we have actually first concluded the formula for the ideal triangles, then for the ones with two vertices at the absolute!).

8 Problems Class 8: Computations in hyperboloid model

5 March 2024

Question 8.1. Given a right-angled triangle with sides a, b, c and corresponding angles α, β, γ, where $\gamma = \pi/2$, show that

$$\sinh a = \sinh c \sin \alpha.$$

<u>*Solution:*</u> We will represent hyperbolic plane by vectors (in $\mathbb{R}^{2,1}$, with pseudo-scalar product $\langle u, v\rangle = u_1v_1 + u_2v_2 - u_3v_3$). Moreover, we will take the representative of each point in the plane $z = 1$, in other words, in the Klein disc model.

By transitivity of isometries on the points of hyperbolic plane, we may assume that the vertex C of the triangle (with the right angle in it) coincides with the centre of the disc (i.e. is represented by vector $v_C = (0, 0, 1)$. Also, after rotating around the z-axis, we can assume that the other two vertices are $v_A = (x, 0, 1)$ and $v_B = (0, y, 1)$. The pseudo-orthogonal vectors to the sides of the triangle (with respect to the considered pseudo-scalar product) are $u_b = (-1, 0, 0)$, $u_a = (0, -1, 0)$ and $u_c = (\frac{1}{x}, \frac{1}{y}, 1)$, as shown in Fig. 19 (check that these vectors indeed define lines containing the corresponding pairs of vertices!)

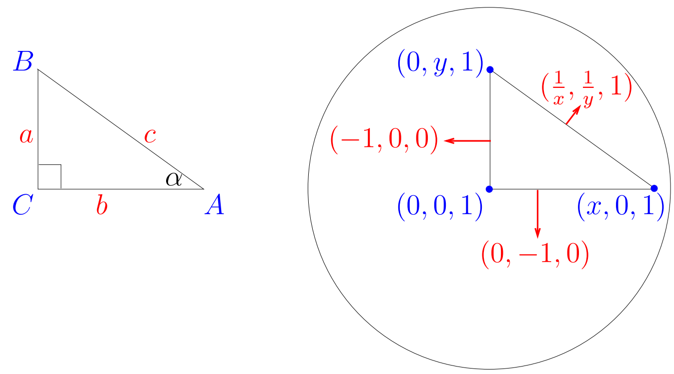

Figure 19: Right-angled triangle in the Klein disc and corresponding vectors.

Now, we will compute all elements applying Theorems 7.3 and 7.4:

- $\sinh^2 a = \left|\frac{(\langle v_B, u_b\rangle)^2}{\langle v_B, v_B\rangle\langle u_b, u_b\rangle}\right| = \left|\frac{y^2}{1\cdot(y^2-1)}\right| = \frac{y^2}{1-y^2}$.
- $\cosh^2 c = \left|\frac{(\langle v_A, v_B\rangle)^2}{\langle v_A, v_A\rangle\langle v_B, v_B\rangle}\right| = \frac{1}{(x^2-1)(y^2-1)}$, from where we conclude

  $\sinh^2 c = \frac{1}{(x^2-1)(y^2-1)} - 1 = \frac{x^2+y^2-x^2y^2}{(x^2-1)(y^2-1)}$.
- $\cos^2 \alpha = \left|\frac{(\langle u_b, u_c\rangle)^2}{\langle u_b, u_b\rangle\langle u_c, u_c\rangle}\right| = \frac{1}{y^2}\frac{1}{(x^2y^2)(x^2+y^2-x^2y^2)}$, from where we have

  $\sin^2 \alpha = 1 - \cos^2 \alpha = \frac{x^2+y^2-x^2y^2-x^2}{x^2+y^2-x^2y^2} = \frac{y^2(1-x^2)}{x^2+y^2-x^2y^2}$.

Combining all three results together we get the statement.

Question 8.2. Prove Pythagorean theorem $\cosh c = \cosh a \cosh b$ for a right-angled triangle with $\gamma = \pi/2$.

Solution: We compute all three entries by Theorem 7.3:

- $\cosh^2 a = \left|\frac{(\langle v_B, v_C\rangle)^2}{\langle v_B, v_B\rangle\langle v_C, v_C\rangle}\right| = \frac{1}{1-y^2}$.
- $\cosh^2 b = \left|\frac{(\langle v_A, v_C\rangle)^2}{\langle v_A, v_A\rangle\langle v_C, v_C\rangle}\right| = \frac{1}{1-x^2}$.
- $\cosh^2 c = \left|\frac{(\langle v_A, v_B\rangle)^2}{\langle v_A, v_A\rangle\langle v_B, v_B\rangle}\right| = \frac{1}{(1-x^2)(1-y^2)}$.

Combining this three results we obtain the statement.

Question 8.3. For a right-angled triangle with $\gamma = \pi/2$ prove $\tanh b = \tanh c \cos \alpha$.

Solution: This follows from the computation done in two previous questions:

- $\frac{\sinh^2 b}{\cosh^2 b} = \frac{\cosh^2 b - 1}{\cosh^2 b} = \frac{1/(1-x^2)\ -1}{1/(1-x^2)} = x^2$;
- $\frac{\sinh^2 c}{\cosh^2 c} = x^2 + y^2 - x^2y^2$;
- $\cos^2 \alpha = \frac{x^2}{x^2+y^2-x^2y^2}$,

which implies the statement.

Question 8.4. Let $ABCD$ be a hyperbolic quadrilateral, with $\angle A = \angle B = \angle C = \pi/2$, $AB = a$ and $BC = b$. Denote $\angle D = \varphi$. Find $\cos \varphi$.

Solution: We place vertex B to the centre of the Klein disc model. Then the quadrilateral is represented by a Euclidean rectangle (as hyperbolic right angle with a line represented by a diameter is also a Euclidean right angle). See Fig. 20 for the diagram and for the vectors representing the vertices and the normals to the sides.

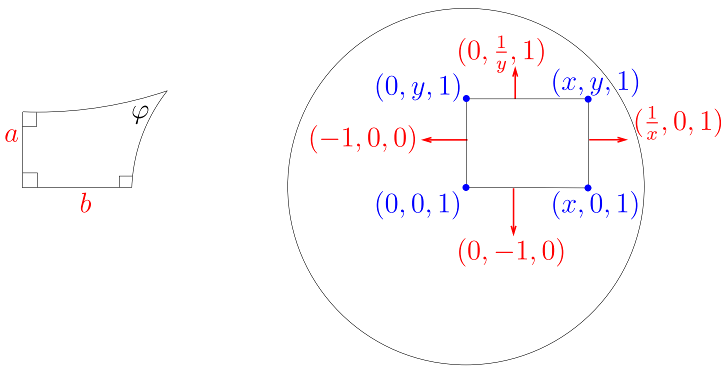

Figure 20: A quadrilateral with three right angles in the Klein disc.

From this (together with Theorem 7.4) we see that

$$\cos^2 \varphi = \frac{0 \cdot \frac{1}{x} + \frac{1}{y} \cdot 0 + 1 \cdot 1}{(\frac{1}{y^2} - 1)(\frac{1}{x^2} - 1)} = \frac{x^2y^2}{(1-x^2)(1-y^2)}.$$

Also, from the same Theorem 7.4 we have

$$\sinh^2 a = \frac{y^2}{1-y^2} \qquad \text{and} \qquad \sinh^2 b = \frac{x^2}{1-x^2}.$$

From this we conclude that

$$\cos\varphi = \sinh a \sinh b.$$

Question 8.5. Let l a be a line in $\mathbb{H}^2$, e be an equidistant curve to the line. Let $N, N \in l$ and $A \in e$. Consider the points $B = e \cap AM$ and $C = e \cap AN$. Show that the area S_{ABC} of triangle $\triangle ABC$ does not depend on the choice of $A \in e$.

Idea of Solution: Drop the perpendiculars AH, BP and CQ from the points A, B, C to l. Then the following two pairs of congruent triangles will arise: $\triangle BMP \cong \triangle AMN$ and $\triangle CNQ \cong \triangle ANH$ (by symmetry with respect to M and N respectively). This implies that $S_{ABC} = S_{BCQP}$, where the later quadrilateral $BCQP$ is a quadrilateral with a side PQ lying on l and having length $2MN$, and points B and C obtained from P and Q by drawing perpendiculars to l till the intersection with the equidistant curve. This only depend on the distance between M and N.

Notice, that the above reasoning only works well for the case when H, the foot of the perpendicular AH, lies between M and N. When it lies outside of of MN, one still can reglue the triangles to form a quadrilateral, but the picture will be a bit more complicated.

Question 8.6. Find the radius of the circle inscribed into an ideal hyperbolic triangle.

Solution: This was Question 15.1(b) in the assignment, and was done by many students using the upper half-plane model. Below, I want to demonstrate that we can do the same also in the Klein model (understood as embedded into $\mathbb{R}^{2,1}$).

By triple transitivity of isometries on the points of the absolute, we may assume that the vertices of the triangle correspond to a regular Euclidean triangle, i.e are given by vectors $(\frac{\sqrt{3}}{2}, -\frac{1}{2}, 1)$, $(0,1,1)$, $(-\frac{\sqrt{3}}{2}, -\frac{1}{2}, 1)$, with the centre of the inscribed circle being the centre of the model, i.e. $O = (0,0,1)$ The point were the inscribed circle is tangent to the "horizontal" side is the middle of the side (again, from the symmetry). i.e. $M = (0, -\frac{1}{2}, 1)$. So, the radius r satisfies

$$\cosh^2 r = \frac{0 \cdot 0 + 0 \cdot (-\frac{1}{2}) + 1 \cdot 1}{1 \cdot (\frac{1}{4} - 1)} = \frac{4}{3},$$

so, $\cosh r = \frac{2}{\sqrt{3}}$ and $r = arcosh\frac{2}{\sqrt{3}}$.

9 Revision

What to expect in the Exam?

- The questions will be on problem solving, not on reproducing the known proofs or following algorithms to get the answer.
- In your solutions you can use any statement proved in the course (i.e. in the lecture notes, Problems classes, or Assignments).
- For the Additional Reading question, you can use any statement proved in the assigned additional reading.

How to prepare?

- Review the material (use the tables!)
- Solve problems! (If you solved most of the questions in the Assignments - not only the marked part of them - you should be in a very good position to get a very good result in your exam).

Now, we are moving to the most important question. The one in the title of a whole book by by George Pólya:

How to solve it?

0. **Draw a diagram!** It will help you to understand the question. It will help to get to some idea of solution. It will help me to get your idea. It will help you to clearly write down the solution!
1. **Choose the geometry:** $S^2, \mathbb{E}^2, \mathbb{H}^2, \mathbb{RP}^1, \mathbb{RP}^2, M\ddot{o}b, Aff$**?**
   When the question is understood, the first job is to figure out which geometry will help to solve it. It is not always necessary the one mentioned in the text of the question. The global task is to find the largest group of transformations preserving all properties described in the question - and use this group to simplify the situation.

   Exercise 9.1. Let $ABCD$ be a trapezoid on Euclidean plane. Let M and N be the midpoints of the parallel sides BC and DA respectively. Let $P = AB \cap CD$ and $Q = AC \cap BD$. Show that the points M, N, P, Q are collinear.

   Solution: The question mentions Euclidean plane, but one can notice that all properties mentioned in the question (collinearity, incidence, parallelism, midpoint) are notions of affine geometry (i.e. preserved by all affine maps). So, we can use affine transformations (which gives us more freedom than Euclidean isometries).

   We can map the triangle $\triangle ADP$ to an isosceles triangle by an affine transformation f (in view of transitivity of affine transformations on the triples of points of the plane), see Fig. 21. Then, as affine maps preserve the ratios of lengths on a line, $f(N)$ is the midpoint of the regular triangle $f(ADP)$, and PN is mapped to an axis of a symmetry of the triangle. Denote by r the reflection with respect to $f(PN)$. As affine maps also preserve parallelism, $f(BC)$ is parallel to $f(AD)$, and hence r maps the line $f(BC)$ to itself. This implies that r swaps the points

$f(B)$ and $f(C)$. In particular, the midpoint $f(M)$ of the segment $f(BC)$ lies on $f(PN)$. Also, from the same symmetry it is clear that the point $f(Q)$ lies on the the line $f(PN)$ (as the reflection with respect to $f(PN)$ swaps $f(B)$ with $f(C)$ and $f(A)$ with $f(D)$). This implies that the points $f(M), f(N), f(P), f(Q)$ are all colliner. Since affine transformations preserve collinearity (and f^{-1} is an affine transformation) we conclude that the original points M, N, P, Q are also collinear.

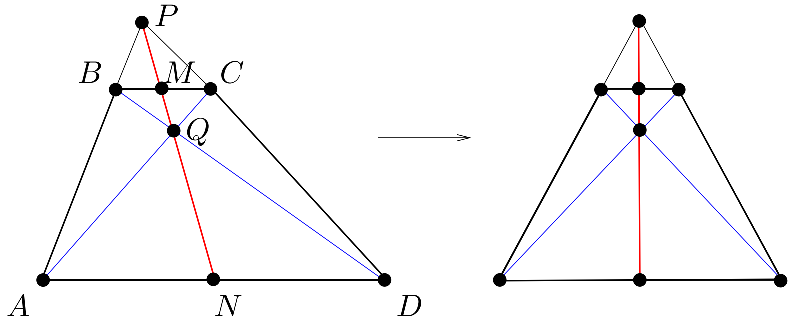

Figure 21: Applying an affine map to simplify the question about the trapezoid.

Remark. One can ask whether we can use the group of projective transformations instead of affine ones (then we could map the quadrilateral to a square and the statement whould be even more obvious!) But the problem is that projective transformations do not preserve the ratios of the distances on the line - so we have no idea where to map the midpoints of the sides. So, we cannot use projective transformations here.

Hint: How to choose the group?
Look at the notions mentioned in the question:

- Lines, collinearity, incidence, concurrence – think of $\mathbb{RP}^2$.
- The same <u>and</u> parallelism or ratios of segments on a line – think of affine transformations.
- Circles and lines, angles, orthogonality – think of Möbius transformations.

2. **Use the group to simplify.** Now, when the appropriate space and the group acting on it are choosen, we need to use the action of the group to simplify the question (so that all properties are preserved!).

Exercise 9.2. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two intersecting circles. Suppose they are not tangent. Let $\mathcal{C}_3$ be a circle orthogonal to both $\mathcal{C}_1$ and $\mathcal{C}_2$. And let $\mathcal{C}_4$ and $\mathcal{C}_5$ be circles tangent to all of $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ (touching the same segment of $\mathcal{C}_3$). Show that $\mathcal{C}_4 \cap \mathcal{C}_5 \neq \emptyset$.

<u>Solution.</u> As we see many circles and some orthogonality, we suppose that the good choice of geometry is the Möbius one. Where to map? It would be good to turn some of the circles into the lines, so makes sense to send some intersection point to ∞. Let O be one of the intersection points of $\mathcal{C}_1$ and $\mathcal{C}_2$. Consider a

Möbius transformation f taking O to ∞. Then $\mathcal{C}_1$ and $\mathcal{C}_2$ are mapped to two intersecting lines (by mapping another intersection point to the origin, we may assume that these are lines through the origin), and $\mathcal{C}_3$ is mapped to a line or circle orthogonal similtaneously to two intersecting lines, so $f(\mathcal{C}_3)$ is a circle centred at the origin (see Fig. 22). Now, it is easy to see that $f(\mathcal{C}_4)$ and $f(\mathcal{C}_5)$ have a common point lying on $f(\mathcal{C}_3)$ (more precisely, from the symmetry with respect to the angle bisector of $\angle f(\mathcal{C}_1)f(\mathcal{C}_2)$ we conclude that both $f(\mathcal{C}_4)$ and $f(\mathcal{C}_5)$ toach $f(\mathcal{C}_3)$ at the middle of the corresponding arc).

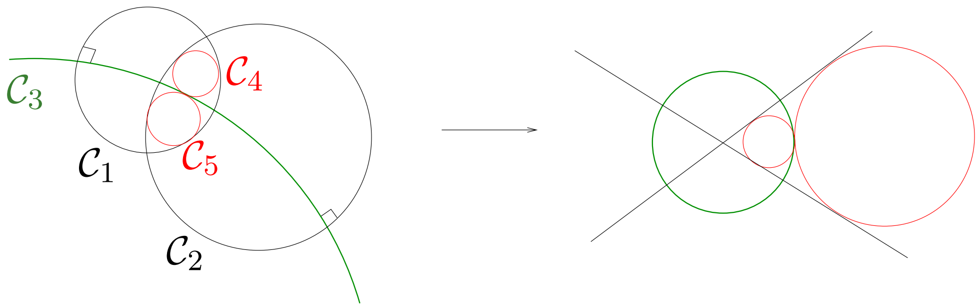

Figure 22: Applying a Möbius map to simplify the question about the circles.

3. **When in $\mathbb{H}^2$ - choose a model.** If the question is about hyperbolic plane, we need first to choose the model we are going to use (before starting to apply the group in attempt to simplify the question).

   There will be no exam question requiring to solve it in any specific model (but the more models you know, the better choice you have!)

   Hint: How to choose the model? This depends on what you see in the question.

   1. Circle, regular polygon, rotation – try Poincaré disc.
   2. Triangle with a vertex at the absolute, angle of parallelism, parallel lines, parabolic isometry, horocycle, equidistant curve – try upper half-plane.
   3. Lines, orthogonality, incidence – try Klein disc.
   4. Computations - try either upper half-plane or hyperboloid model.
   5. Maybe, you do not need any model at all, but will be able to derive the required properties from the statements we have already shown in the course?

4. **Solve the question.** ... Often easier to say than to do...
   To find the solution (or the plan for your solution) you could try to look at the following list.

   Tools:

   a. Formula sheet.
   b. In $S^2, \mathbb{E}^2, \mathbb{H}^2$:
      - congruence of triangles;

- angle sum, area;
- sine/cosine laws;
- triangle inequality.

c. In S^2: polarity of triangles.

d. In $\mathbb{RP}^1$, $\mathbb{RP}^2$, $M\ddot{o}b$, $\mathbb{H}^2$: cross-ratio.

e. In $\mathbb{H}^2$: angle of parallelism.

5. **Nothing helps?** Return to steps 3, or 2, or even 1...
   Remember that all questions in the exam have reasonably short solutions (your lecturers have solved them and typed the solutions to submit the exam, and they are quite lazy to do something long - so, it should be also possible for you!)

6. **Write** and **illustrate** your solutions! Underline the answer.

7. **Sanity check.** Have you obtained negative area or length? Does the answer conctradict something we know (say, some existence or uniqueness theorem)?

Questions about isometries:

There are several types of questions about isometries (or other transformation groups in various geometries). This may include:

- What type of an element is some f? (f may be given as a composition of other elements).
- For a given element f, is it true that $f = id$?
- Find an image of some object under some map (may be given as a composition of some other maps).

Hints:

- Write everything through the generators of the group:
  For $S^2, \mathbb{E}^2, \mathbb{H}^2$ consider reflections;
  For $M\ddot{o}b$ consider Möbius transformations together with anti-Möbiuse ones and look at inversions and reflections.
- Use non-uniqueness of presentation of an element as a composition of generators - to simplify the things.
- Remember (or check in the table!) the classification of group elements.
- Consider the fixed points of the elements.

Example. Let γ be a circle in $\mathbb{H}^2$. Suppose that $f(\gamma) = \gamma$ for some isometry $f \in Isom(\mathbb{H}^2)$. Describe f.

Solution. Let O be the centre of γ. Then $f(O) = O$. Hence, f has a fixed point inside $\mathbb{H}^2$. Consider the classification of isometries in $\mathbb{H}^2$ (identity, reflection, elliptic, parabolic, hyperbolic, glide reflection). Since f has a fixed point, we conclude that it is either identity, or a reflection (with respect to any line containing O), or an elliptic isometry, i.e. a rotation around O by any angle.

Example. Let R by an anticlockwise rotation around the origine by angle $\pi/2$ on Euclidean plane. Let T be a troanslation by 1 (given by $z \to z+1$). Describe the composition $f = T \circ R$.

Solution. $R = r_2 \circ r_1$ where r_1 is the reflection with respect to $\{(x,y) \in \mathbb{R}^2 \ x = y\}$, and r_2 is the reflection with respect to $\{(x,y) \in \mathbb{R}^2 \ x = 0\}$. Similarly, $T = r_3 \circ r_2$, where r_3 is the reflection with respect to $\{(x,y) \in \mathbb{R}^2 \ x = 1/2\}$ and r_2 as before. So,

$$f = T \circ R = r_3 \circ r_2 \circ r_2 \circ r_1 = r_3 \circ r_1.$$

This implies that f is the rotation about the intersection point of the lines $y = 2$ and $x = y$, i.e. a rotation about $(1/2, 1/2)$ (by angle $\pi/2$, in clockwise direction).

Questions on finding an invariant quantity.

Example. Let $G = \langle g_1, g_2 \rangle$, where $g_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $g_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ act on $\mathbb{R}^2$ as linear transformations. Let $A = (0,0), B = (1,0), C = (0,1)$.

(a) Is there an element $g \in G$ such that $g(\triangle ABC) = \triangle ACB$?
Answer: No, since such a map g would change the **orientation** of the triangle, while both generators g_1 and g_2 preserve the orientation (due to positive determinants).

(b) Is there an element $g \in G$ such that $g(\triangle ABC) = \triangle ACD$, where $D = (0,3)$?
Answer: No, since such a map would change the **area** of the itrangle, while both g_1 and g_2 preserve the area (due to unit determinant - or, alternatively - due to direct check of these elements).
Here we use that given a matrix M and triangle Δ we have

$$Area(M\Delta) = |\det M| Area(\Delta).$$

(c) Is there an element $g \in G$ such that $g(\triangle ABC) = \triangle AFE$, where $E = (1,1)$, $F = (2,1)$?
Answer: Yes, $f = g_1 \circ g_2$.

Remark: The answer will be "yes" here for any triangle of unit area and same orientation a $\triangle ABC$, as g_1 and g_2 are the generators of $SL_2(\mathbb{Z})$. It is sufficient to demonstrate and example producing f (show your work!).

Hint: Possible invariants: length, angles, area, cross-ratio, orientation, collinearity....

How to check collinearity?

(a) For a triple of points in $\mathbb{E}^2$ or $\mathbb{E}^3$:
points $p, q, r \in \mathbb{R}^3$ are collinear if and only if $p - q = k(p - r)$ for some $k \in \mathbb{R}$ (here, $p = (p_1, p_2, p_4)$ and similarly for q and r).

(b) For a triple of points in $\mathbb{RP}^2$:
three points in $\mathbb{RP}^2$ are collinear if and only if the corresponding three lines in $\mathbb{R}^3$ are coplanar. We may assume that the three lines are given by vectors $u, v, w \in \mathbb{R}^3$. Then these vectors are coplanar if and only if $w = k_1 u + k_2 v$. In homogenious coordinates we can state $w_i = k_1 u_i + k_2 v_i$ for $i = 1, 2, 3$.

Example. Points $u = (1,2,3)$, $v = (2,3,4)$ and $w = (4,7,10)$ are collinear as $w = 2u + v$.

(c) The points $z_1, z_2, z_3 \in \mathbb{C}$ are collinear if and only if $\frac{z_2 - z_1}{z_3 - z_1} \in \mathbb{R}$.

(d) The points $z_1, z_2, z_3, z_4 \in \mathbb{C}$ lie on one line or circle if $\frac{z_3 - z_1}{z_3 - z_2} \Big/ \frac{z_4 - z_1}{z_4 - z_2} \in \mathbb{R}$.

Finally, thank you for your participation in Geometry module.
I wish you all possible success in the exams and afterwards in your lifes!