Durham University
Anna Felikson

Geometry 2023/2024

**Questions for Problems classes**

Here are some questions which will be probably discussed in the Problems Class (subject to change!)

# 1 Reflections on the plane, Geometric constructions

## Problems Class 1 (17 October, 2023)

1. Let $R_{A,\varphi}$ and $R_{B,\psi}$ be rotations with $0 < \varphi, \psi \leq \pi/2$. Find the type of the composition $f = R_{B,\psi} \circ R_{A,\varphi}$.

   *Hint:* This is an example of using reflections to study compositions of isometries (write everything as a composition of reflections, make you choice so that some of them cancel!).

2. Let $A$ and $B$ be two given points in one half-plane with respect to a line $l$. How to find a shortest path, which starts at $A$ then travels to $l$ and returns to $B$? (How to find the point where this path will reach the line $l$?)

3. Ruler and compass constructions: perpendicular bisector, perpendicular from a point to a line, midpoint of a segment, angle bisector, inscribed and circumscribed circles for a triangle.

# 2 Group actions on $\mathbb{E}^2$

## Problems Class 2 (31 October, 2023)

0. Let $g_1, \dots, g_n$ be isometries of $\mathbb{E}^2$. Let $G = \langle g_1, \dots, g_n \rangle$ be the group generated by $g_1, \dots, g_n$ (i.e. the minimal group containing all of $g_1, \dots, g_n$). Show that the group $G$ acts on $\mathbb{E}^2$.

1. Let $G$ be a group generated by two reflections on $\mathbb{E}^2$. When $G$ is discrete?

2. Let $T$ be a triangle with angles $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}$. Let $r_1, r_2, r_3$ be the reflection with respect to the sides of $T$, and let $G$ be the group generated by $r_1, r_2, r_3$. In the lecture we have checked that $G : \mathbb{E}^2$ discretely. Find the fundamental domain of this action.

3. Find the orbit-space for the action introduced in Question 2.

4. Let $X$ be a regular triangle on $\mathbb{E}^2$. Let $r_1$ and $r_2$ be two distinct reflections taking $X$ to itself. Find the fundamental domain of the action $G : X$. Find also the orbit-space.

5. Let $G$ be a group generated by rotation through angle $\frac{2\pi}{3}$ on the plane. Find the orbit-space of the action $G : \mathbb{E}^2$. Are there closed geodesics in this orbit-space? Are there bi-infinite open geodesics?

# 3 Spherical Geometry

### Problems Class 3 (14 November, 2023)

1. Let $G : S^2$ be an action. $G$ acts discretely if and only if $|G| < \infty$.
2. Let $G : X$ be an action and suppose that $F$ is its fundamental domain. Then one can show that the action $G : X$ is discrete.
3. Let $g$ be a reflection, $h \in Isom(S^2)$. $h$ is a reflection if and only if there exists $f \in Isom(S^2)$ such that $fgf^{-1} = h$.
4. Let $S^2$ be a sphere of radius 1. Show that the length of a circle or(spherical) radius $r$ equals to $2\pi \sin r$.

   **Remark:** for the sphere of radius $R$, the length of the circle of radius $r$ will be $2\pi R \sin(\frac{r}{R})$. When $R \to \infty$ we see that $\frac{r}{R} \to 0$ and, hence, $2\pi R \sin(\frac{r}{R}) \to 2\pi r$.
5. Let $S^2$ be a sphere of radius $R$. Let $\alpha$ and $\beta$ be two parallel planes crossing $S^2$. Find the area of the part of $S^2$ lying between the planes $\alpha$ and $\beta$.
6. One can also discuss ruler and compass constructions, as in $\mathbb{E}^2$.

# 4 Projective geometry

### Problems Class 4 (28 November, 2023)

1. Find a projective transformation $f$ which takes

$$\begin{aligned} A = (1:0:0) &\text{ to } (0:0:1) \\ B = (0:1:0) &\text{ to } (0:1:1) \\ C = (0:0:1) &\text{ to } (1:0:1) \\ D = (1:1:1) &\text{ to } (1:1:1) \end{aligned}$$

   Find the image of $X = AD \cap BC$ under this transformation.
2. Find $[A, B, C, D]$ for the points above. (Does it exist?)
   For $E = (1 : 1 : 0)$, $F = (1 : 2 : 0)$ find $[A, B, E, F]$.
3. Check explicitly, that the transformation $f$ from Question 1 preserves the value of $[A, B, E, F]$.
4. Let $A_1, A_2, A_3, A_4$ be points on a line $a$, let $B_1, B_2, B_3, B_4$ be points on a line $b$. Denote by $p_i$ the line through $A_i$ and $B_i$. Show that if the lines $p_1, p_2, p_3, p_4$ are concurrent, then the points $A_{i+1}B_i \cap A_iB_{i+1}$ $(i = 1, 2, 3)$ are collinear.
5. Formulate and prove the statement dual to the one in Question 4.

# 5 Möbius geometry

## Problems Class 5 (23 January, 2024)

1. Find the type of Möbius transformation $f(z) = 1/z$.
2. Let $f$ and $g$ be inversions with respect to two intersecting circles. Show that $g \circ f = f \circ g$ if and only if $Fix_f \perp Fix_g$.
3. Let $\mathcal{C}_1, \dots, \mathcal{C}_5$ be circles all passing through the points $A$ and $B$ on the plane. Show that there exists a circle $\gamma$ orthogonal to all of $\mathcal{C}_i$.
4. Prove Ptolemy's theorem: given a quadrilateral $ABCD$ inscribed into a circle, one has

$$|AB| \cdot |CD| + |BC| \cdot |AD| = |AC| \cdot |BD|.$$

# 6 Poincaré disc

## Problems Class 6 (6 February, 2024)

1. Let $l_1$ and $l_2$ be two divergent lines. Show that there exists a unique common perpendicular to $l_1$ and $l_2$.
2. Let $0 \leq \alpha, \beta, \gamma < \pi$, $\alpha + \beta + \gamma < \pi$. Then there exists a hyperbolic triangle with angles $\alpha, \beta, \gamma$.
3. Show that every triangle in $\mathbb{H}^2$ has an inscribed circle.
4. Given a hyperbolic triangle, is it true that it always has a circumscribed circle?
5. Ruler and compass constructions in hyperbolic plane:
   - midpoint of a segment;
   - perpendicular bisector;
   - angle bisector;
   - centre of a given circle;
   - tangent line to a given circle;
   - centre of inscribed circle for a triangle;
   - centre of circumscribed circle for a triangle (when exists);
   - common perpendicular for two divergent lines (in assumptions that one is given an infinitely long ruler, which allows to draw lines through two points of the absolute).
6. A polygon with all vertices at the absolute is called *ideal*. Is it true that all ideal hyperbolic quadrilaterals are isometric to each other?

# 7 Some computations in hyperbolic plane

### Problems Class 7 (20 February, 2024)

1. Show that area of hyperbolic disc of radius $r$ is $4\pi \sinh^2(\frac{r}{2})$.
2. [Hyperbolic oranges] Consider hyperbolic oranges, i.e. a discs of radius $r$, where the inner disc of radius $\frac{9}{10}r$ is a pulp, while the outer $1/10$ is a peal. Assuming that all oranges are equally tasty, which oranges are better to buy: big or small?
3. [Hyperbolic golf] Assume you are playing golf on hyperbolic field and your aim is 300m apart from you. You made a very good shoot, by sending a ball 300m far away with only $1^o$ mistake. How far away from you goal will the ball land?
4. We had many statements about hyperbolic triangles (e.g. sine and cosine rules, congruence theorems, area formula) - which of them still hold for unbounded triangles, having at least one vertex at the absolute?

# 8 Computations in hyperboloid model

### Problems Class 8 (5 March, 2024)

1. Given a right-angled triangle with sides $a, b, c$ and corresponding angles $\alpha, \beta, \gamma$, where $\gamma = \pi/2$, show that
$$\sinh a = \sinh c \sin \alpha.$$
2. Prove Pythagorean theorem $\cosh c = \cosh a \cosh b$ for a right-angled triangle with $\gamma = \pi/2$.
3. For a right-angled triangle with $\gamma = \pi/2$ prove $\tanh b = \tanh c \cos \alpha$.
4. Let $ABCD$ be a hyperbolic quadrilateral, with $\angle A = \angle B = \angle C = \pi/2$, $AB = a$ and $BC = b$. Denote $\angle D = \varphi$. Find $\cos \varphi$.
5. Find the radius of the circle inscribed into an ideal hyperbolic triangle.
6. Let $l$ a be a line in $\mathbb{H}^2$, $e$ be an equidistant curve to the line. Let $N, N \in l$ and $A \in e$. Consider the points $B = e \cap AM$ and $C = e \cap AN$. Show that the area $S_{ABC}$ of triangle $\triangle ABC$ does not depend on the choice of $A \in e$.