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Christmas Problems

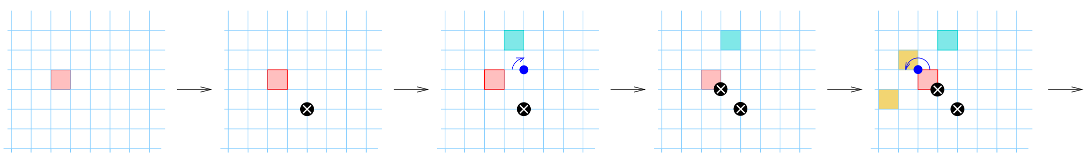
Snowed problems to be sent to Santa Claus
by Sunday, 24 December

1. (*) **Hope and Hate.** **Angel of Hope** and **Troll of Hate** play the game on infinite grid of square cells. The rules are as follows:

   - At every move, Angel of Hope is allowed to choose any (unfrozen) vertex V of the grid and a rotation R_V about V preserving the grid. Then all squares obtained by R_V from previously coloured squares become also coloured.
   - At every move, Troll of Hate is allowed to freeze any vertex of the grid forever.
   - Before the first move, there is exactly one coloured square.
   - Before the first move, every vertex of the grid is unfrozen.
   - The first move belongs to Troll of Hate, and then the moves alternate.
   - Angle of Hope wins if after infinitely many steps all squares of the board are coloured. And looses otherwise...

   Will Angle of Hope be able to win if both are playing best possible way?

Example:

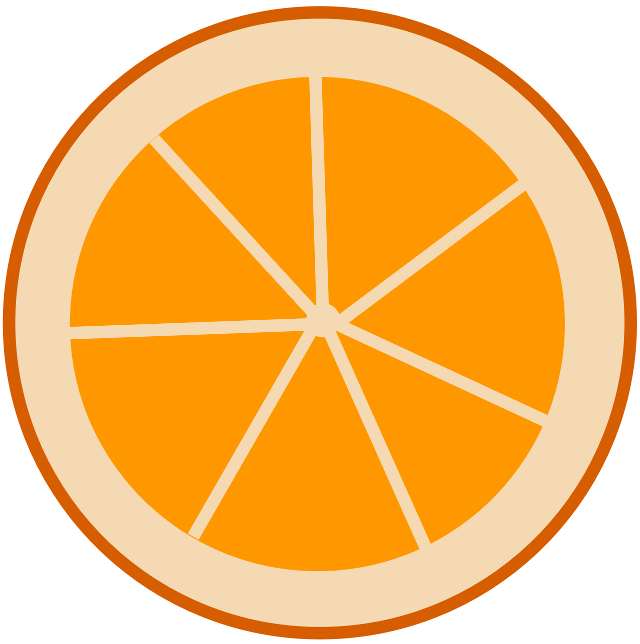


2. Buying spherical oranges.

A spherical orange is a disc on a sphere. These oranges have quite thick peel: an orange of radius r has only a $\frac{4}{5}r$-disc of pulp and the rest is the peel. Oranges are good and tasty, the peel is rubbish.

a) A shop sells big and small oranges (both are tasty) for the same price per unit of total area. What costs less per unit of area of pulp, the big ones or the small ones?

b) The shop also sells Euclidean oranges for the same price.
Which would you prefer to buy, spherical or Euclidean ones?



3. Cutting a triangular cake.

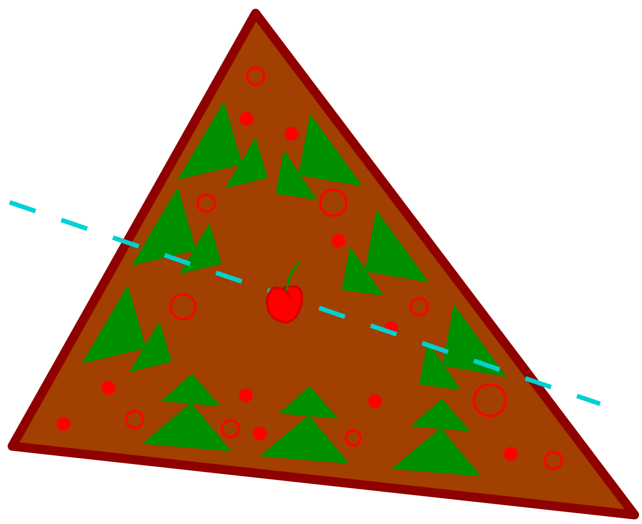
Alice and Bob have a triangular cake to share. Alice chooses a point A inside the cake, then Bob chooses a straight line through A to cut the cake in two parts. Alice gets the **smaller** part and Bob the bigger.

(a) Where should Alice place the point to get as much of the cake as possible?

(b) Now, modify the rules so that Alice gets the **bigger** part. Show that then Bob can always get exactly a half (if he could guess the right line l).

(c) In the settings of (b), can you propose an explicit algorithm allowing Bob to construct a line l which cuts the cake almost into two halves (i.e. as close to the two halves as Bob wants)?

(d) The same questions if the cake is a spherical triangle. (**Warning:** this question is difficult!)




HAPPY
NEW
YEAR!