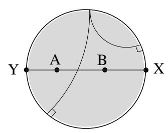
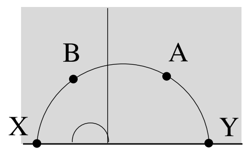
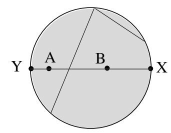
Anna Felikson, Durham University 

# Models of hyperbolic geometry

| model | Poincaré disk | Upper half-plane | Klein disk | two-sheet hyperboloid |
|---|---|---|---|---|
| $\mathbb{H}^2$ | $\{z \in \mathbb{C} \mid \lvert z\rvert < 1\}$ | $\{z \in \mathbb{C} \mid Imz > 0\}$ | $\{z \in \mathbb{C} \mid \lvert z\rvert < 1\}$ | $\{v \in \mathbb{R}^{2,1} \mid \langle v, v\rangle = -1,\ v_3 > 0\}$<br>where $\langle v, u\rangle = v_1u_1 + v_2u_2 - v_3u_3$ |
| $\partial\mathbb{H}^2$ (absolute) | $\{z \in \mathbb{C} \mid \lvert z\rvert = 1\}$ | $\{z \in \mathbb{C} \mid Imz = 0\}$ | $\{z \in \mathbb{C} \mid \lvert z\rvert = 1\}$ | $\{v \in \mathbb{R}^{2,1} \mid \ \langle v, v\rangle = 0,\ v_3 > 0\}$<br>$v \sim \lambda v$ |
| lines |  |  |  | $\{v \mid \langle v, a\rangle = 0\}$<br>where $\langle a, a\rangle > 0$ |
| distance | $d(A,B) = \lvert ln[A,B,X,Y]\rvert$<br>$X, Y =$ the "endpoints" of the line $AB$ | | $d(A,B) =$<br>$\frac{1}{2}\lvert ln[A,B,X,Y]\rvert$ | $d(A,B) = \frac{1}{2}\lvert ln[A,B,X,Y]\rvert$<br>cross-ratio of four lines* |
| formula | | $\cosh d(u,v) =$<br>$1 + \frac{\lvert u-v\rvert^2}{2Im(u)Im(v)}$ | | $Q = \lvert\frac{\langle u,v\rangle^2}{\langle u,u\rangle\langle v,v\rangle}\rvert$<br>if $\langle u,u\rangle < 0$, $\langle v,v\rangle < 0$<br>$Q = \cosh^2 d(pt,pt)$<br>if $\langle u,u\rangle < 0$, $\langle v,v\rangle > 0$<br>$Q = \sinh^2 d(pt, line)$<br>if $\langle u,u\rangle > 0$, $\langle v,v\rangle > 0$<br>$Q < 1$, intersecting lines<br>$Q = \cos^2\alpha$<br>$Q = 1$, parallel lines<br>$Q > 1$, ultraparallel lines<br>$Q = \cosh^2 d(line, line)$ |
| isometries** | Möbius transformations | | Projective tr | Linear transformations of $\mathbb{R}^{2,1}$ |
| orientation-preserving isometries | | $\frac{az+b}{cz+d}$<br>$a,b,c,d \in \mathbb{R},\ \ ad-bc = 1$ | | |
| orientation-reversing isometries | | $\frac{a\bar{z}+b}{c\bar{z}+d}$<br>$a,b,c,d \in \mathbb{R},\ \ ad-bc = -1$ | | |
| reflections | Euclidean inversions or reflections | | | $r_a(v) = v - 2\frac{\langle v,a\rangle}{\langle a,a\rangle}a$ |
| circles | Euclidean circles | | ellipses | plane sections of the hyperboloid |
| angles | angles=Euclidean angles | | distorted angles<br>good for right angles*** | |

* Cross-ratio of four lines lying in one plane and passing through one point is the cross-ratio of four points at which these lines are intersected by an arbitrary line $l$ (it does not depend on $l$!).

**We only list the type of the transformations not specifying that they preserve the model.

*** See the next page.

***Right angles in the Klein model**.
Let $l$ be a hyperbolic line.
Let $\bar{l}$ be a Euclidean line containing the segment which represents $l$ in the Klein model.
Let $X_1(l)$ and $X_2(l)$ be the endpoints of $l$ (intersections of $\bar{l}$ with the unit circle).
Let $t_1(l)$ and $t_2(l)$ be tangent lines to the unit circle at the points $X_1(l)$ and $X_2(l)$.
Let $T(l) = t_1(l) \cap t_2(l)$ (if $t_1 || t_2$, i.e. $l$ is represented by a diameter, then $T(l)$ is a point at infinity).

**Thm.** $l'$ is orthogonal to $l$ if and only if $T(l) \in l'$.

In particular, if $l$ is represented by a **diameter**, then $l' \perp l$ if and only if $\bar{l}' \perp \bar{l}$ (in Euclidean sinse).