Riemannian Geometry

Hints 13-14

1. For (a) use that $[fX, Y] = f[X, Y] - (Yf)X$.
2. (*) Write the definition of the curvature tensor and regroup the terms to get Lie brackets.
3. (a) First, compute Christoffel symbols, the $\nabla_X Y$ for $X, Y \in \{\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\theta}\}$, then compute R by definition.
4. Step 1. Show that the expression (v_1, v_2, v_3, v_4) is multilinear.

   Step 2. Show that (v_1, v_2, v_3, v_4) has the same symmetries as Riemannian curvature tensor.

   Step 3. Show that if $\{v_1, v_2, v_3, v_4\} \subset \{v, w\}$, i.e. no more than two distinct vectors are involved, then
   $$\langle R(v_1, v_2)v_3, v_4\rangle = (v_1, v_2, v_3, v_4).$$

   Step 4. Show that if no more than three distinct vectors are involved, then
   $$\langle R(v_1, v_2)v_3, v_4\rangle = (v_1, v_2, v_3, v_4).$$

   Step 5. Show that for any four vectors $\{v_1, v_2, v_3, v_4\}$
   $$\langle R(v_1, v_2)v_3, v_4\rangle - (v_1, v_2, v_3, v_4) = \langle R(v_3, v_1)v_2, v_4\rangle - (v_3, v_1, v_2, v_4),$$
   i.e. the difference above is invariant with respect to cyclic permutation of first three arguments.

   Step 6. Use Bianchi identity to prove the initial statement.
5. Compute Christoffel symbols, then use it to compute $\nabla_{\frac{\partial}{\partial_i}} \frac{\partial}{\partial_j}$, and find K and R by definition.
6. Same plan as above.
7. (a) Reconstruct symmetric bilinear form $Ric_p(v, w)$ by the quadratic form $Ric_p(v)$.