Riemannian Geometry, Epiphany 2014.

Homework 11-12

Starred problems due on Friday, February 7th

1. (*) Consider the upper half-plane $M = \{(x, y) \in \mathbf{R}^2 \mid y > 0\}$ with the metric
$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{y} \end{pmatrix}$$
   (a) Show that all the Christoffel symbols are zero except $\Gamma^2_{22} = -\frac{1}{2y}$.
   (b) Show that the vertical segment $x = 0$, $\varepsilon \le y \le 1$ with $0 < \varepsilon < 1$ is a geodesic curve when parametrized proportionally to arc length.
   (c) Show that the length of the segment $x = 0$, $\varepsilon \le y \le 1$ with $0 < \varepsilon < 1$ tends to 2 as ε tends to zero.
   (d) Show that (M, g) is not geodesically complete.

2. (a) Show that
$$\mathrm{Exp}\left(t \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 1 & t & t^2/2 & t^3/(3!) \\ 0 & 1 & t & t^2/2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
   Guess how the answer would be for the Lie group exponential of a $n \times n$-matrix of the same form (i.e., only entries 1 at the first upper diagonal).
   (b) Use the fact (you don't need to prove this) that if A, B commute then
$$\mathrm{Exp}(A)\mathrm{Exp}(B) = \mathrm{Exp}(A + B),$$
   in order to show that
$$\mathrm{Exp}\left(t \begin{pmatrix} c & 1 & 0 & 0 \\ 0 & c & 1 & 0 \\ 0 & 0 & c & 1 \\ 0 & 0 & 0 & c \end{pmatrix} \right) = e^{tc} \begin{pmatrix} 1 & t & t^2/2 & t^3/(3!) \\ 0 & 1 & t & t^2/2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

3. (a) Let $H_3(\mathbf{R})$ be a set of 3×3 upper triangular matrices
   (i.e. the matrices of the form $\begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix}$, where $x_1, x_2, x_3 \in \mathbf{R}$).
   Show that the set $H_3(\mathbf{R})$ form a group. This group is called the Heisenberg group.
   (b) Show that the Heisenberg group is a Lie group. What is its dimension?
   (c) Prove that the matrices
$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
   form a basis for the tangent space $T_e H_3(\mathbf{R})$ of the group $H_3(\mathbf{R})$ at the neutral element e.
   (d) For each $k = 1, 2, 3$, find an explicit formula for the curve $c_k : \mathbf{R} \to SL_2(\mathbf{R})$ given by $c_k(t) = Exp(tX_k)$.

4. Let $(G, \langle \cdot, \cdot \rangle)$ be a *compact* Lie group with left-invariant metric and let *dvol* denote the corresponding left-invariant volume form. Compactness implies that $vol(G) < \infty$ (you don't need to prove this). Define an inner product $\langle\langle \cdot, \cdot \rangle\rangle_e$ at $e \in G$ by

$$\langle\langle v_1, v_2 \rangle\rangle_e := \int_G \langle Ad(g^{-1})v_1, Ad(g^{-1})v_2 \rangle_e \, dvol(g),$$

and let $\langle\langle \cdot, \cdot \rangle\rangle_g$ denote the left-invariant extension to a Riemannian metric on G. Show that $\langle\langle \cdot, \cdot \rangle\rangle_g$ is a bi-invariant Riemannian metric on G:

(a) Show first that
$$\langle\langle Ad(h^{-1})v_1, Ad(h^{-1})v_2 \rangle\rangle_e = \langle\langle v_1, v_2 \rangle\rangle_e$$
for all $h \in G$, by using the fact that left-invariance of *dvol* implies that
$$\int_G f(L_h(g)) \, dvol(g) = \int_G f(g) \, dvol(g).$$
(You may use this fact without proof.)

(b) Use the fact $Ad(h^{-1}) = DL_{h^{-1}}(h) DR_h(e)$ in order to show
$$\langle\langle DR_h(e)v_1, DR_h(e)v_2 \rangle\rangle_h = \langle\langle v_1, v_2 \rangle\rangle_e \quad \text{for all } h \in G,$$
i.e., the right-invariance of $\langle\langle \cdot, \cdot \rangle\rangle_g$.

Remark: The above averaging procedure is called the *Weyl trick*.

5. (*) Let $(G, \langle \cdot, \cdot \rangle)$ be a Lie group with a *bi-invariant* Riemannian metric. Let $\mathfrak{g}$ denote the corresponding Lie algebra of left-invariant vector fields on G. Our aim is to show for $X, Y \in \mathfrak{g}$ that
$$\nabla_X Y = \frac{1}{2}[X, Y].$$

(a) Use the relation
$$\langle Z, \nabla_X Y \rangle =$$
$$\frac{1}{2} \left(X\langle Z, Y \rangle + Y\langle Z, X \rangle - Z\langle Y, X \rangle + \langle X, [Z, Y] \rangle + \langle Y, [Z, X] \rangle - \langle Z, [Y, X] \rangle \right)$$
and the fact that the metric is left-invariant to prove that $\langle Z, \nabla_Y Y \rangle = \langle Y, [Z, Y] \rangle$ for $X, Y, Z \in \mathfrak{g}$.

(b) By Corollary 7.17, the bi-invariance of the metric implies that
$$\langle [U, X], V \rangle = -\langle U, [V, X] \rangle$$
for $X, U, V \in \mathfrak{g}$. Use this fact in order to conclude that $\nabla_Y Y = 0$ for all $Y \in \mathfrak{g}$.

(c) Show that $\nabla_X Y = \frac{1}{2}[X, Y]$.

6. As in the lecture, let G be a Lie group, $H \subset G$ be a closed subgroup, $\pi : G \to G/H$ be the canonical projection, $\langle \cdot, \cdot \rangle_e$ be an $Ad(H)$-invariant inner product on T_eG, $V \subset T_eG$ be the orthogonal complement to $T_eH \subset T_eG$ with respect to $\langle \cdot, \cdot \rangle_e$, and Φ the restriction of $D\pi(e) : T_eG \to T_{eH}G/H$ to the subspace V. Prove the following statements:

(a) $T_eH = \ker D\pi(e)$.
(You may use without proof that $D\pi(e) : T_eG \to T_{eH}G/H$ is surjective.)

(b) $\Phi : V \to T_{eH}G/H$ is an isomorphism.

(c) V is $Ad(H)$-invariant.
(**Hint:** The fact that $Ad(h_1)Ad(h_2) = Ad(h_1h_2)$ might be useful.)