Riemannian Geometry, Epiphany 2014.

Homework 13-14

Starred problems due on Friday, February 21th

1. Let (M, g) be a Riemannian manifold and R its curvature tensor. Let $f, g, h \in C^\infty(M)$, and X, Y, Z, W be vector fields on M. Show that
   (a) $R(fX, Y)Z = fR(X, Y)Z$;
   (b) $R(X, fY)Z = fR(X, Y)Z$;
   (c) $\langle R(X, Y)fZ, W\rangle = \langle fR(X, Y)Z, W\rangle$;
   (d) $R(fX, gY)hZ = fghR(X, Y)Z$.

2. (*) Let (M, g) be a Riemannian manifold and R its curvature tensor. Prove the *First Bianchi Identity*:
   $$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$$
   for X, Y, Z vector fields on M by reducing the equation to Jacobi identity.

3. (*) Parametrize the sphere S^2 of radius r in $\mathbf{R}^3$ by
   $$(x, y, z) = (r\cos\varphi\sin\theta,\ r\sin\varphi\sin\theta,\ r\cos\theta),$$
   and consider the metric induced from Euclidean metric in $\mathbf{R}^3$ on it.
   (a) Compute $R(\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\theta})$.
   (b) Compute sectional curvature of the sphere S^2 of radius r.

4. Let (M, g) be a Riemannian manifold. Show that M is of constant sectional curvature K_0 if and only if
   $$\langle R(v_1, v_2)v_3, v_4\rangle = K_0(\langle v_1, v_4\rangle\langle v_2, v_3\rangle - \langle v_1, v_3\rangle\langle v_2, v_4\rangle)$$
   for any $p \in M$ and $v_1, v_2, v_3, v_4 \in T_pM$.
   Hint: see hints for the detailed plan of the solution.

5. **Constant sectional curvature of hyperbolic 3-space**
   (a) Show that sectional curvature $K(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3})$ and $K(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$ of 3-dimensional hyperbolic space $\mathbf{H}^3$ (in upper half-space model) is equal to -1.
   (b) Use (a) and linearity of Riemannian curvature tensor to show that 3-dimensional hyperbolic space $\mathbf{H}^3$ has constant sectional curvature.

6. **Horosphere in hyperbolic 3-space**
   Let $\mathbf{H}^3$ be the upper half-space model of the hyperbolic 3-space.
   Consider a *horosphere*
   $$M = \{x \in \mathbf{H}^3 \mid x_1^2 + x_2^2 + (x_3 - 1)^2 = 1\}$$
   in hyperbolic 3-space with metric g induced from $\mathbf{H}^3$.
   (a) Parametrize M using spherical coordinates, and compute the induced metric.
   (b) Compute the Christoffel symbols of (M, g).
   (c) Compute the curvature tensor of (M, g). More precisely, prove that the curvature tensor is identically zero.

7. (*) A Riemannian manifold (M, g) is called *Einstein manifold* if there exists $c \in \mathbf{R}$ such that

$$Ric_p(v, w) = c\langle v, w \rangle$$

for every $p \in M$, $v, w \in M_p$.

(a) Show that (M, g) is Einstein manifold if and only if there exists $c \in \mathbf{R}$ such that

$$Ric_p(v) = c$$

for every $p \in M$ and unit tangent vector $v \in M_p$.

(b) Show that if (M, g) is of constant sectional curvature then (M, g) is Einstein manifold.