Riemannian Geometry, Epiphany 2014.

Homework 15 -16

Starred problems due on Friday, March 7th

1. (*)

   (a) Let $M \subset \mathbf{R}^n$ be a Riemannian manifold with a metric induced from $\mathbf{R}^n$. Let $p, q \in M$ be two points. Show that
   $$d_M(p,q) \geq d_{\mathbf{R}^n}(p,q),$$
   where $d_M(p,q)$ is the distance from p to q in M, and $d_{\mathbf{R}^n}(p,q)$ is the distance between the same points in $\mathbf{R}^n$.

   (b) Recall that Bonnet-Myers theorem implies that if (M,g) is complete, and there is $\varepsilon > 0$ such that $Ric_p(v) > \varepsilon$ for every $p \in M$ and for every unit tangent vector v, then the diameter of M is finite.
   Use the surface $M = \{(x,y,z) \in \mathbf{R}^3 \mid x^2 + y^2 + 1 = z^2\}$ to show that the assumption $\varepsilon > 0$ is essential (i.e. can not be substituted by the assumtion $Ric_p(v) > 0$).
   Hint: parametrize M by $(x,y,z) = (r\cos\varphi, r\sin\varphi, \sqrt{r^2+1})$.

2. **Second Variation Formula of Energy**
   Let $F : (-\varepsilon, \varepsilon) \times [a,b] \to M$ be a proper variation of a geodesic $c : [a,b] \to M$, and let X be its variational vector field. Let $E : (-\varepsilon,\varepsilon) \to \mathbf{R}$ denote the associated energy, i.e.,
   $$E(s) = \frac{1}{2}\int_a^b \|\frac{\partial F}{\partial t}(s,t)\|^2 dt.$$
   Show that
   $$E''(0) = \int_a^b \|\frac{D}{dt}X\|^2 - \langle R(X,c')c', X\rangle dt$$

3. (*) Let $S^2 = \{x \in \mathbb{R}^3 \mid {x_1}^2 + {x_2}^2 + {x_3}^2 = 1\}$ be a unit sphere, and $c : [-\pi/2, \pi/2] \to S^2$ be a geodesic defined by $c(t) = (\cos t, 0, \sin t)$. Define a vector field $X : [-\pi/2,\pi/2] \to TS^2$ along c by
   $$X(t) = (0, \cos t, 0).$$
   Let $\frac{D}{dt}$ denote covariant derivative on S^2 along c.

   (a) Calculate $\frac{D}{dt}X(t)$ and $\frac{D^2}{dt^2}X(t)$.

   (b) Show that X satisfies the Jacobi equation.

4. **Jacobi fields on manifolds of constant curvature.**
   Let M be a Riemannian manifold of constant sectional curvature K, and $c : [0,1] \to M$ be a geodesic satisfying $\|c'\| = 1$. Let $J : [0,1] \to TM$ be a orthogonal Jacobi field along c (i.e. $\langle J(t), c'(t)\rangle = 0$ for every $t \in [0,1]$).

   (a) Show that $R(J,c')c' = KJ$.
   Hint: You may use the result of Problem 4 from HW13-14.

   (b) Let $Z_1, Z_2 : [0,1] \to TM$ be parallel vector fields along c with $Z_1(0) = J(0)$, $Z_2(0) = \frac{DJ}{dt}(0)$. Show that
   $$J(t) = \begin{cases} \cos(t\sqrt{K})Z_1(t) + \frac{\sin(t\sqrt{K})}{\sqrt{K}}Z_2(t) & \text{if } K > 0, \\ Z_1(t) + tZ_2(t) & \text{if } K = 0, \\ \cosh(t\sqrt{-K})Z_1(t) + \frac{\sinh(t\sqrt{-K})}{\sqrt{-K}}Z_2(t) & \text{if } K < 0. \end{cases}$$
   Hint: Show that these fields satisfy Jacobi equation, there value and covariant derivative at $t = 0$ is the same as for $J(t)$, and then use uniqueness (Corollary 10.5).