Riemannian Geometry, Epiphany 2014.

Homework 17 -18

Starred problems due on Tuesday, March 18th

1. (*) Let M be a Riemannian manifold of non-positive sectional curvature.

   (a) Let $c : [a, b] \to M$ be a differentiable curve and J be a Jacobi field along c. Let $f(t) = \|J(t)\|^2$. Show that $f''(t) \geq 0$, i.e., f is a convex function.

   (b) Derive from (a) that M does not admit conjugate points.

2. (a) Let $c(t)$ be a geodesic, and $c(t_0)$ is conjugate to $c(t_1)$. Let J be any Jacobi field along c vanishing at t_0 and t_1. Show that J is normal, i.e. $\langle J(t), c'(t)\rangle \equiv 0$.

   (b) Show that the dimension of the space $J_c^{\perp}$ of normal vector fields along c is $2n - 2$.

3. (*) Let $c : [0, 1] \to M$ be a geodesic, and J be a Jacobi field along c.
   Denote $c(0) = p, c'(0) = v$.

   Define a curve $\gamma(s)$,

$$\gamma : (-\varepsilon, \varepsilon) \to M, \qquad \gamma(0) = p, \gamma'(0) = J(0)$$

   Define also a vector field $V(s) \in T_\gamma M$, such that

$$V(0) = v, \qquad \frac{D}{ds}V(0) = \frac{D}{dt}J(0),$$

   and a variation $F(s, t) = exp_{\gamma(s)} tV(s)$.

   (a) Show that $F(s, t)$ is a geodesic variation of $c(t)$.

   (b) Show that $\frac{\partial F}{\partial s}(0, 0) = \gamma'(0) = J(0)$, and $\frac{D}{dt}\frac{\partial F}{\partial s}(0, 0) = \frac{D}{ds}V(0) = \frac{D}{dt}J(0)$.

   (c) Deduce from (a) and (b) that every Jacobi field along a geodesic $c(t)$ is a variational vector field of some geodesic variation of c.

4. Let $M = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 = z\}$ be a paraboloid of revolution with metric induced from $\mathbf{R}^3$. Let $p = (0, 0, 0)$. Show that p has no conjugate points in M.