Riemannian Geometry, Epiphany 2014.

7 Crash course: Basics about Lie groups

A Lie group is a smooth manifold wit a smooth group structure (see Definition 1.12.)

Examples: matrix Lie groups: $GL(n,\mathbf{R})$, $SL(n,\mathbf{R})$, $O(n)$, $SO(n)$.

7.1 Left-invariant vector fields and Lie algebra

Definition 7.1. Let G a be a Lie group, $g \in G$. Then the maps $L_g : G \to G$ and $R_g : G \to G$ defined by $L_g(h) = gh$ and $R_g(h) = hg$ are diffeomorphisms of G called left- and right-translation.

Remark: 1. $L_{g^{-1}} \circ L_g = id_G$, $\quad L_{g_1} R_{g_2}(h) = R_{g_2} L_{g_1}(h) = g_1 h g_2$.
2. The differential $DL_g : T_hG \to T_{gh}G$ gives a natural identification of tangent spaces.
Moreover, $DL_g : \mathfrak{X}(G) \to \mathfrak{X}(G)$ defines a map of vector fields by the following formula:
$(DL_gX)(h) := DL_g(g^{-1}h)(X(g^{-1}h))$.

Definition 7.2. A vector field $X \in \mathfrak{X}(G)$ is called left-invariant if $DL_gX = X$ for all $g \in G$.

Remark 7.3. 1. Left-invariant vector fields on G form a linear space over $\mathbf{R}$.
2. Left-invariant vector field is determined by its value at e: if $X(e) = v$ then $X(g) = DL_g(e)v$.
3. Hence, the space of left-invariant vector fields on G may be identified with T_eG.

Definition 7.4. The space of left-invariant vector fields on G is called the Lie algebra of G and denoted by $\mathfrak{g}$.

Lemma 7.5. For arbitrary $X \in \mathfrak{X}(G)$, $f \in C^\infty(G)$ holds $((DL_gX)f)_g = X(f \circ L_g)$.

Corollary 7.7. If X is left-invariant, then $(Xf) \circ L_g = X(f \circ L_g)$ for any $f \in C^\infty(G)$.

Proposition 7.8. Let X be a Lie group with Lie algebra g. Then for any $X, Y \in \mathfrak{g}$ holds $[X,Y] \in \mathfrak{g}$. Consequently, $\mathfrak{g}$ is a Lie algebra in terms of Definition 2.22.

Example 7.9. Computation of $DL_g(e)$ for a matrix Lie group. In case of matrix Lie group, the left-invariant vector field $X \in \mathfrak{g}$ with $X(e) = v$ is given by $X(g) = gv$.

7.2 Lie group exponential map and adjoint representation

Define $Exp : M(n,\mathbf{R}) \to M(n,\mathbf{R})$ by $Exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!}A^k$.

Properties: 1. the infinite sum converges for any matrix $A \in M(n,\mathbf{R})$, so $Exp(A)$ is well-defined.
2. $Exp(0) = e = diag\{1,\dots,1\}$ ($n \times n$ diagonal matrix with 1's on the diagonal).
3. if $AB = BA$ then $Exp(A+B) = Exp(A) \cdot Exp(B)$.
4. $Exp(A) \in GL(n,\mathbf{R})$ for any $A \in M(n,\mathbf{R})$: $(ExpA)^{-1} = Exp(-A)$.

Proposition 7.10. Let G be a matrix Lie group. Let $v \in T_eG$ and let X be a unique left-invariant vector field on G with $X(e) = v$. Then the curve $c(t) = Exp(tv) \in G$ satisfies $c(0) = e$, $c'(0) = v$ and $c'(t) = X(c(t))$.

A curve of the form $c(t) = Exp(tv)$ is called the 1-parameter subgroup in G with $c'(0) = v$.

Example 7.11: computation of the exponent for a diagonalizable matrix.

Remark. In case of abstract Lie group the exponential map may be defined as follows. Let G be a Lie group and g be its Lie algebra. Let $v \in T_eG$ and $X \in \mathfrak{g}$ be a unique left-invariant vector field with $X(e) = v$. Then there exists a unique curve $c : \mathbf{R} \to G$ with $c_v(0) = e$, $c_v'(t) = X(c_v(t))$ [without proof]. The curve c_v is called an integral curve of X. We define the exponential map by $Exp(v) = c_v(1)$.

Definition 7.12. Let G be a Lie group. For $g \in G$ the adjoint representation $Ad(g) : T_eG \to T_eG$ is defined by

$$Ad(g)(w) := \frac{d}{dt}\Big|_{t=0} L_g R_{g^{-1}}(Exp(tw)) = \frac{d}{dt}\Big|_{t=0} gExp(tw)g^{-1}.$$

For $v \in T_eG$ the adjoint representation $ad(v) : T_eG \to T_eG$ is defined by

$$ad(g)(v) := \frac{d}{dt}\Big|_{t=0} Ad(Exp(tv))(w) = \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} Exp(tv)Exp(tw)Exp(-tv).$$

Theorem 7.13. (without proof) Let G be a Lie group with a Lie algebra $\mathfrak{g}$. Then for all $X, Y \in \mathfrak{g}$ holds $[X, Y](e) = ad(X(e))(Y(e)) \in T_eG$, i.e. by canonical identification of $\mathfrak{g}$ with T_eG we have $[X, Y] = ad(X)Y$.

Example 7.14. Theorem 7.13 for the case of a matrix Lie group.

7.3 Riemannian metrics on Lie groups

Definition 7.15. For a given inner product $< \cdot, \cdot >_e$ on T_eG, define the inner product at $g \in G$ for $v, w \in T_eG$ by $< v, w >_g := < DL_{g^{-1}}(g)v, DL_{g^{-1}}(g)w >_e$. The family $(< \cdot, \cdot >_g)_{g \in G}$ of inner products defines a left-invariant Riemannian metric on G. (Every left-invariant Riemannian metric is obtained this way).

Remark 7.16. Let $(G, <,>)$ be a Lie group with a left-invariant metric. Then
1. the diffeomorphisms $L_g : G \to G$ are isometries;
2. for any two left-invariant vector fields $X, Y \in \mathfrak{g}$ the map $g \mapsto < X(g), Y(g) >_g$ is a constant function.

Theorem 7.17. Let G be a compact Lie group. Then G admits a biinvariant Riemannian metric $< \cdot, \cdot >_g$, i.e. both families of diffeomorphisms L_g and R_g are isometries.

Corollary 7.18. Let $(G, <,>)$ be a Lie group with biinvariant metric. Then for $X, Y, Z \in \mathfrak{g}$ holds $< [X, Y], Z >= - < Y, [X, Z] >$.

Corollary 7.19. Let $(G, <,>)$ be a Lie group with biinvariant metric and let ∇ be the Levi-Civita connection. Then for $X, Y \in \mathfrak{g}$ holds $\nabla_X Y = \frac{1}{2}[X, Y]$.

Remark 7.20. The 1-parameter subgroups are exactly the geodesics of the left-invariant metric on G. So, the Lie group exponential map Exp coincides with the Riemannian exponential map exp_p at the identity.

7.4 Invariant metric on homogenious spaces **(supplemetary matherial, not for exam!)**

Definition 7.21. A connected Riemannian manifold (M, g) is called homogeneous if the group of isometries of M acts transitively on M.

Examples: $\mathbf{E}^n$, S^n, $\mathbf{H}^n$.

Construction: Given a Lie group G and a closed subgroup $H \subset G$, consider the set $M = G/H = \{gH \mid g \in G\}$. Then
1. M is a smooth manifold (non-trivial theorem, uses that H is closed);
2. there is a canonical projection $\pi : G \to G/H$, $\pi(g) = gH$;
3. The elements of G act on M by $\widetilde{L}_g(g'H) = gg'H$, the diffeomorphism $\widetilde{L}_g : M \to M$ is called left G-action.

If there is a left-invariant metric on M, (i.e. $< D\widetilde{L}_g(e)v, D\widetilde{L}_g(e)w >_{gH} = < v, w >_{eH}$ $\forall g \in G$, $v, w \in T_{eH}(G/H)$) then $\widetilde{L}_g$ is an isometry of M, and M is a homogeneous space.

Theorem 7.22. (without proof). Each homogeneous manifold may be obtained in this way.

Theorem 7.23. (without proof)
The left-invariant metrics $< \cdot, \cdot >$ on G/H are in one-to-one correspondence to $Ad(H)$-invariant inner products $< \cdot, \cdot >_e$ on T_eG, i.e. $< Ad(h)v, Ad(h)w >_e = < v, w >_e$ for all $h \in H$, $w, v \in T_eG$.

Example. $G = SO(3)$, isometries of a sphere S^2; $H = SO(2)$ isometries of the sphere stabilizing one point, $M = SO(3)/SO(2) = S^2$, the sphere.
More generally, for a homogeneous space M one may coinside the group of its isometries $Isom(M)$ with a subgroup $Stab_p \subset Isom(M)$ of isometries stabilizing one point $p \in M$. Then, one can prove $M = Isom(M)/Stab_p$.

8 Curvature

8.1 Riemannian curvature tensor

Definition 8.1. Let (M, g) be a Riemannian manifold, let $\mathfrak{X}(M)$ be vector field on M, and let ∇ be a Levi-Civita connection. Define a map (Riemannian curvature tensor) $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ by $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_{Y X} Z \nabla_{[X,Y]} Z$.

Remark: R is linear in all variables; in particular, $R(fX, gY)hZ = fghR(X,Y)Z$ (so, it is a tensor).

Lemma 8.2. R has the following symmetries:

(a) $R(X,Y)Z = -R(Y,X)Z$

(b) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ (first Bianchi Identity)

(c) $\langle R(X,Y)Z, W\rangle = -\langle R(X,Y)W, Z\rangle$

(d) $\langle R(X,Y)Z, W\rangle = -\langle R(Z,W)X, Y\rangle$

Remark: Define components of Riemannian curvature tensor by $R_{ijkl} = \langle R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\rangle$, and define R^l_{ijk} by $R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})\frac{\partial}{\partial x_k} = \sum_l R^l_{ijk}\frac{\partial}{\partial x_l}$.
Then $R_{ijkl} = \sum_m R^l_{ijk} g_{ml}$ and $R^l_{ijk} = \sum_m R_{ijkm} g^{ml}$.

Remark: In $\mathbf{E}^n$, we have $R \equiv 0$.

Definition 8.3. A Riemannian manifold is called flat if it is locally isometric to $\mathbf{E}^n$ (i.e. each point has a neighbourhood isometric to an open set in $\mathbf{E}^n$).

Theorem 8.4. (without proof)
A Riemannian manifold is flat if and anly if its Riemannian curvature tensor vanishes identically.

Example 8.5. Let G be a Lie group with a biinvariant metric.
Then $R(X,Y)Z = -\frac{1}{4}[[X,Y],Z]$ for all $X, Y, Z \in \mathfrak{g}$.

Example 8.6: components R_{ijks} and R^l_{ijk} for hyperbolic plane (in the upper half-plane model).

8.2 Sectional curvature

Definition 8.7. Let (M, g) be a Riemannian manifold, let $p \in M$ be a point, let $v_1, v_2 \in T_pM$ be tangent vectors and $\Pi \subset T_pM$ be a 2-plane spanned by v_1, v_2.
The sectional curvature of Π at p is $K(\Pi) = K(v_1, v_2) = \frac{\langle R(v_1,v_2)v_2, v_1\rangle}{||v_1||^2||v_2||^2 - \langle v_1, v_2\rangle^2}$.

Proposition 8.8. $K(\Pi)$ does not depend on the basis $\{v_1, v_2\}$ of Π.

Examples 8.9-8.10: Sectional curvature of hyperbolic 3-space is -1;
sectional curvature of a paraboloid of revolution is positive.

8.3 Ricci and scalar curvature

Given $v, w \in T_pM$ define a linear map $R(\cdot, v)w : T_pM \to T_pM$ by $u \mapsto R(u, v)w$.

Definition 8.11. Ricci curvature tensor $Ric(v, w)$ is the trace of the map $R(\cdot, v)w$: $Ric_p(v, w) = tr(R(\cdot, v)w)$.
In orthonormal basis $\{u_i\}$, $Ric_p(v, w) = \sum_{j=1}^n \langle R(u_j, v)w, u_j\rangle$.
Ricci curvature at p is $Ric_p(v) = Ric_p(v, v) = \sum_{j=1}^n \langle R(u_j, v)w, u_j\rangle$
In orthonormal basis $\{v = u_1, \dots, u_n\}$ we have $Ric_p(v) == \sum_{j=2}^n K(v, u_j)$.

Lemma 8.12. $Ric(v, u)$ is a symmetric bilinear form (i.e. $Ric(v)$ is a quadratic form).

Example 8.13. If $K(v, w)$ is constant ($= K$) for vectors with $||v|| = 1$, then $Ric(v) = (n-1)K$.

Definition 8.14. Scalar curvature $s(p) = \sum_j Ric_p(u_j, u_j)$.

Example: If $K(v, w)$ is constant ($= K$) then $s = n(n-1)K$.

Lemma 8.15. $s(p)$ does not depend on the orthonormal basis $\{u_j\}$.

9 Bonnet-Myers Theorem

Theorem 9.1. (**Second variation formula of length**).
Let $c : [a, b] \to M$ be a geodesic, let $F : (-\varepsilon, \varepsilon) \times [a, b] \to M$ be a proper variation, let $X(t) = \frac{\partial F}{\partial s}(0, t)$ be a variation vector field. Define $X^{\perp}(t) = X(t) - \langle X(t), c'(t)\rangle c'(t)$, the orthogonal component of $X(t)$. Let $l(s)$ be the length of variation.
Then $l''(0) = \int_a^b (||\frac{DX^{\perp}}{dt}||^2 - K(c', X^{\perp})||X^{\perp}||^2)dt$.

Remark: In case if $X^{\perp}$ is collinear to c' (i.e. $X^{\perp} = 0$) we define $K(c', X^{\perp}) := 0$.

Corollary 9.2. If $K(\Pi) < 0$ for each $p \in M$ and each 2-plane $\Pi \in T_pM$ (space of negative curvature) then every geodesic is minimal.

Theorem 9.3. (Bonnet-Myers, 1935) Let (M, g) be connected, complete Riemannian manifold of dimension n. Suppose that $Ric(v) \geq \frac{n-1}{r^2}$ for all $v \in SM = \{w \in TM \mid ||w|| = 1\}$.
Then $diamM := sup_{p,q \in M} d(p, q) \leq \pi r$. In particular, M is bounded, so, it is compact (as it is complete).

Example 9.4. For n-dimensional sphere S^n_r of radius r the inequality in the Bonnet-Myers Theorem turns into equality. Hence, the bound is sharp.

Example 9.5. Let $T^n = \mathbf{R}^n/\mathbf{Z}^n$ be an n-dimensional torus with arbitrary metric g. Then there is no $c > 0$ such that $Ric(v) \geq c$ for all $p \in M$, $v \in T_pM$ (otherwise the lift of g to $\mathbf{R}^n$ contradicts to B-M Theorem).

10 Jacobi fields

10.1 Jacobi fields and geodesic variations

Definition 10.1. Let $c(t)$ be a geodesic. A vector field $J \in \mathfrak{X}_c(M)$ is a Jacobi field if it satisfies Jacobi equation: $\frac{D^2}{dt^2}J + R(J, c')c' = 0$.

Example 10.2. Vector fields $c'(t)$ and $tc'(t)$ are Jacobi fields for any geodesic $c(t)$.

Theorem 10.3. Let $c(t)$ be a geodesic. Let $F(s, t)$ be a variation, s.t. every curve $F_s(t)$ is geodesic. Then the variation field $X(t) = \frac{\partial F}{\partial s}(0, t)$ is Jacobi field.

Example 10.4. Geodesic variation on a sphere and its variation field.

Definition 10.5. Let $E_1(t), \dots, E_n(t) \in \mathfrak{X}_c(M)$ be vector fields on $c(t)$. We say that $\{E_, \dots, E_n\}$ is a parallel orthonormal basis along c if for all t, i, j holds $\frac{D}{dt}E_i = 0$ and $< E_i, E_j >= \delta_{ij}$.

Notation: $R_{ij} =< R(E_i, c')c', e_j >$, $\quad n \times n$ symmetric matrix depending on t.

Theorem 10.6. Let $c(t)$ be a geodesic and $\{E_i\}$ be a parallel orthonormal basis along c. Take $J \in \mathfrak{X}_c(M)$ and its expansion $J = \sum_j J_j(t)E_j(t)$ (where $J_j(t)$ is a function). Then J is a Jacobi field is and only if $J''_k + \sum_{k=1}^n R_{kj}J_j = 0$ for all $k = 1, \dots, n$.

Corollary 10.7. For any choice of $v, w \in T_{c(t_0)}M$ there exists a unique Jacobi field J along c such that $J(t_0) = v$, $\frac{D}{dt}J(t_0) = w$.

Remark 10.8. Corollary 10.7 implies that for any geodesic $c(t)$ there is a $2n$-dimensional space $J_c(M)$ of Jacobi fields on c. Moreover, the map $T_{c(t_0)}M \times T_{c(t_0)}M \to J_c(M)$ defined by $(v, w) \mapsto J$ s.t. $J(t_0) = v$, $\frac{D}{dt}J(t_0) = w$ is an isomorphism.

Lemma 10.9. Let $c : [0, 1] \to M$ be a geodesic and $J \in J_c(M)$ be a Jacobi field along c. Suppose $J(0) = 0$. Then there exists a geodesic variation F of c such that $J = \frac{\partial F}{\partial s}(0, t)$.

10.2 Conjugate points and normal Jacobi fields

Definition 10.10. Let $c : [a, b] \to M$ be a geodesic, $a \leq t_0 < t_1 \leq b$ and $p = c(t_0)$, $q = c(t_1)$ be two points. The point q is conjugate to p along $c(t)$ if there exists a Jacobi field $J \in J_c(M)$, $J \neq 0$ such that $J(t_0) = J(t_1) = 0$.

Example 10.11. On the sphere S^2 (with induced metric), the South pole is conjugate to the North pole along each geodesic passing through both these points.

Definition 10.12. A point $q \in M$ is conjugate to a point $p \in M$ if there exists a geodesic $c(t)$ passing through p and q such that q is conjugate to p along $c(t)$.

Definition 10.13. A multiplicity of a conjugate point $c(t_1)$ (with respect to the point $c(t_0)$ is the number of linear independent Jacobi fields along c such that $J(t_0) = J(t_1) = 0$, in other words, it is $dim J_c^{t_0,t_1}(M)$ where $J_c^{t_0,t_1}(M) = \{J \in J_c(M) \mid J(t_0) = J(t_1) = 0\}$.

Remark 10.14. Multiplicity does not exceed $n - 1$.

Lemma 10.15. Let $J \in J_c(M)$ be a Jacobi field along a geodesic $c(t) = exp_p tv$. Suppose $J(0) = 0$. Then there exists $v, w \in T_{c(0)}M$ s.t. $J(t) = (Dexp_p)_{tv} tw$.

Lemma 10.16. The point $q = c(t_1)$ is conjugate to $p = c(0)$ along a geodesic $c(t) = exp_p tv$ if and only if the point $v_1 = t_1 v$ is a critical point of the exponential map exp_p (i.e. $dim Ker(Dexp_p)_{t_1 v} > 0$). Multiplicity of q is equal to $dim Ker(Dexp_p)_{t_1 v}$.

Lemma 10.17. Let $c : [a, b] \to M$ be a geodesic, $a \le t_0 < t_1 \le b$. Suppose that $c(t_1)$ is not conjugate to $c(t_0)$. Take $v \in T_{c(t_0)}M$, $u \in T_{c(t_1)}M$. Then there exists a unique Jacobi field J along c s.t. $J(t_0) = v$, $J(t_1) = u$.

Lemma 10.18. Let $J \in J_c(M)$ be a Jacobi field along a geodesic $c(t)$. Then the function $t \mapsto < J(t), c'(t) >$ is linear. Namely, $< J(t), c'(t) > = < J(0), c'(0) > + t < \frac{D}{dt}\big|_{t=0} J(t), c'(0) >$.

Corollary 10.19. Let $< J(t_1), c'(t_1) > = < J(t_2), c'(t_2) >$. Then $< J(t), c'(t) > = const$, a constant function.

Definition 10.20. A Jacobi field $J \in J_c(M)$ is normal if $< J, c' > \equiv 0$.
Notation: $J_c^\perp := \{J \in J_c(M) | < J, c' > \equiv 0\}$.

Corollary 10.21. (1) Let $J(0) = 0$. Then J is normal if and only if $< \frac{D}{dt}\big|_{t=0} J(t), c'(0) > = 0$.
(2) $dim J_c^{\perp, t_0} = n - 1$ where $J_c^{\perp, t_0} := \{J \in J_c(M) | < J, c' > \equiv 0, J(t_0) = 0\}$.
(3) $dim J_c^\perp = 2n - 2$.

Example 10.22. Jacobi fields on $\mathbf{R}^2$.

Theorem 10.23. Let c be a geodesic. Then every Jacobi field $J \in J_c(M)$ is a variation field for some geodesic variation $F(s, t)$ of c.

10.3 Minimizing geodesics and conjugate points

Theorem 10.24. Let $c : [0, b] \to M$ be a geodesic and let $c(a)$ be a point conjugate to $c(0)$, $0 < a < b$. Then c is not minimal between $c(0)$ and $c(b)$.
Lemma 10.25, Corollary 10.26 and Lemma 10.27 serve to prove Theorem 10.24; we skip it here.

Examples 10.28, 10.29: Jacobi fields on the sphere and hyperbolic plane.

10.4 Minimizing geodesics and conjugate points

Definition 10.24. A topological space is simply connected if for each curve $:[0, 1] \to M$ with $c(0) = c(1)$ there exists a continuous map $F : [0, 1] \times [0, 1] \to M$ such that $F(1, t) = c(t)$, $F(0, t) = p$ for some $p \in M$.

Examples: $\mathbf{R}^n$ is simply connected, S^n is simply connected for $n > 1$; S^1 and T^n (torus) are not simply-connected.

Theorem 10.31. (Cartan-Hadamard). Let M be a complete connected, simply connected Riemannian manifold of non-positive sectional curvature. Then M is diffeomorphic to $\mathbf{R}^n$, where n is the dimension of M.

11 Appendix: Curvature and Geometry

The contents of this section is not included in the Examination.
In lectures, the statements were presented without proofs.

11.1 Cut locus

Example 11.1. Flat (Euclidean) torus: no conjugate points, but there are non-minimal geodesics.

Definition 11.2. Let c be a geodesic, $p = c(0)$. A cut point of p with respect to c is $q = c(t_0)$, such that the geodesic c is minimizing on $[0, t_0]$ and not minimizing on $[0, t]$ for $t > t_0$.
A cut locus of p is the set of all cut points of p (with respect to al geodesics though p).

Example 11.3. Cut loci on the sphere S^n and on a flat torus T^2.

Proposition 11.4. If $c(t_0)$ is the cut point of $p = c(0)$ along c, then
(a) either $c(t_0)$ is the first conjugate point of $c(0)$ along c;
(b) or there exists a geodesic $\gamma \neq c$ from p to $c(t_0)$ such that $l(\gamma) = l(c)$.
Conversely, if (a) or (b) holds then there exists $t_1 \in (0, t_0]$ s.t. $C(t_0)$ is a cut point of p along c.

Corollary 11.5. 1) If q is a cut point of p along c then p is a cut point of Q along c.
2) If Q is not a cut point of p along c then there exists a unique minimizing geodesics joining p to q.

11.2 Injectivity radius

Definition 11.6. The injectivity radius of a point $p \in M$ is $i_p := \sup_r \{r \mid exp_p \text{is diffeo in } B_r(p)$.
The injectivity radius of M is $i(M) := \inf_p i_p$.

Proposition 11.7. If M is complete, with sectional curvature K satisfying $0 < K_{min} < K < K_{max}$ then
(a) $i(M) \geq \pi/\sqrt{K_{max}}$;
(b) there exists a shortest closed geodesic $c \in M$ s.t. $i(M) = \frac{1}{2}l(c)$.

11.3 Sphere Theorem

Theorem 11.8. (Sphere Theorem). Let M be a compact, simply connected Riemannian manifold with $\frac{1}{4} < K(\Pi) \leq 1$ for all $\Pi \in T_pM$, for all $p \in M$. Then M is homeomorphic to S^n.

Remark 11.9. 1. Recently, it was proved that M is also diffeomorphic to S^n.
2. The Theorem 11.8 foes not hold for $\frac{1}{4} \leq K(\Pi) \leq 1$.
3. In case of dimension $n = 2$ stronger result holds:
If $K \geq 0$ for all $p \in M$ and $K > 0$ in at least one point, then M is homeomorphic to S^2.

Theorem 11.10. Any smooth manifold of dimension $n \geq 3$ admits a Riemannian metric of negative Ricci curvature.

Remark. The theorem does not hold for surfaces! (Look at S^2 and apply Gauss-Bonnet).

11.4 Spaces of constant curvature

Theorem 11.11. Let M be a complete, simply connected Riemannian manifold of constant sectional curvature K. Then
1) if $K > 0$ then M is isometric to S^n;
2) if $K = 0$ then M is isometric to $\mathbf{E}^n$;
3) if $K < 0$ then M is isometric to $\mathbf{H}^n$.

Remark. If M is not simply connected, then the statement holds locally only.

11.5 Index form

Recall: given a geodesic $c : [0, a) \to M$ there exists a bilinear symmetric form on $\mathfrak{X}_c M$ given by
$I_a(V, W) = \int\limits_0^a (< V, W > + < R(V, c')c', W >)dt.$

Definition: The quadratic form $I_a(V, V)$ is called an index form.

Definition 11.12. The index of I_a is the maximal dimension of a subspace of $\mathfrak{X}_c M$ on which I_a is negative definite.

Theorem 11.11. (Morse Index Theorem). The index of I_a is finite for each geodesic c. Moreover, it equals to the number of points $c(t)$, $0 < t < a$ conjugate to $c(0)$. each counted with its multiplicity.

Corollary 11.12. The set of conjugate points along a geodesic is a discrete set.

Lemma 11.15. (Index Lemma). Let $c : [0, a] \to M$ be a geodesic containing no conjugate points to $c(0)$. Let $J \in J_c$ be a normal Jacobi field, $< J, c' >= 0$. Let V be a piecewise differentiable vector field on c, $< V, c' >= 0$. Suppose also $J(0) = V(0) = 0$, $J(t_0) = V(t_0)$ for some $t_0 \in (0, a]$.
Then $I_{t_0}(J, J) \le I_{t_0}(V, V)$, where equality holds only if $V = J$ on $[0, a]$.

11.6 Comparison Theorems

Theorem 11.16. (Rauch's Comparison Theorem). Let $c : [0, a] \to M^n$ and $\tilde{c} : [0, a] \to \widetilde{M}^{n+k}$, $k \ge 0$ be two unite speed geodesics and let $J : [0, a] \to TM$ and $\widetilde{J} : [0, a] \to T\widetilde{M}$ be normal Jacobi fields along c and $\widetilde{c}$ with $J(0) = 0$, $\widetilde{J}(0) = 0$, $||J'(0)|| = ||\widetilde{J}'(0)||$. Assume that $\widetilde{J}$ does not have conjugate points on $[0, a)$ and that for any $t \in [0, a]$ the inequality $K_M(\Pi) \le K_{\widetilde{M}}(\widetilde{\Pi})$ holds for all 2-planes $\Pi \subset T_{c(t)}M$ and $\widetilde{\Pi} \subset T_{\tilde{c}(t)}\widetilde{M}$. Then $||J(t)|| \ge ||\widetilde{J}(t)||$ for all $t \in [0, a]$.

Example. Regular triangles with side $\pi/2$ in S^2, $\mathbf{E}^2$ and $\mathbf{H}^2$: length of the median.

Definition 11.17. A triangle in a Riemannian manifold is a collection of 3 points with minimal geodesics connecting them. A generalized triangle is a collection of 3 points with any geodesics connecting them and satisfying triangle inequality.

Definition 11.18. A comparison triangle $p'q'r'$ for a generalized triangle $pqr \in M$ is a triangle in a space of constant curvature with sides of of the same lengths.

Remark. In $\mathbf{E}^n$ and $\mathbf{H}^n$ such a triangle always exists.
In S^n it does exist if the lengths in pqr a not too big ($l \le \pi r$, where r is the radius of the sphere).

Theorem 11.19. (Alexandrov-Toponogov Comparison Theorem). Let $K(\Pi) \ge 0$ for all $\Pi \in T_pM$ for all $p \in M$. Let $p_0, p_1, p_2 \in M$. Let p_3 lie between p_1 and p_2 (i.e. $|p_1 - p_3| + |_2 - p_3| = |p_1 - p_2|$). Let p_0', p_1', p_2' be a comparison triangle in $\mathbf{E}^2$. Define p_3' by $|p_i - p_3|_M = |p_i' - p_3'|_{\mathbf{E}^2}$, for $i = 1, 2$.
Then $|p_0 - p_3|_M \ge |p_0' - p_3'|_{\mathbf{E}^2}$ (Alexandrov-Toponogov inequality).
Conversely, if Alexandrov-Toponogov inequality holds for all p_0, p_1, p_2, p_3 then $K \ge 0$.

Remark. 1. Dual statement for $K \le O$ with inverse AT-inequality.
2. Equivalent conditions:
   a. smaller K imply smaller angles;
   b. smaller K imply bigger circumference of a circle of radius r;
   c. smaller K imply bigger volume of a ball or radius r.