Riemannian Geometry, Epiphany 2014.

Standard problems

Integration on (M, g)

- Given a function $f : M \to \mathbf{R}$ integrate it over M (or a over a subset $A \subset M$).

  $\int_M f = \int_V f \circ \varphi^{-1}(x) \sqrt{det(g_{ij}) \circ \varphi^{-1}(x)} dx$, where φ a chart, $V = \varphi(M)$

- Find the volume of M (or of a subset $A \subset M$).

  $VolM = \int_V f \sqrt{det(g_{ij}) \circ \varphi^{-1}(x)} dx$,

Lie groups

- Given a group G, prove it is a Lie group.

  Smoothness of g^{-1} and $g_1 g_2$

- Given a Lie group G and $v \in T_e G$, find the left invariant vector field X such that $X(e) = v$.

  $X(g) = gv$ for matrix Lie groups

- For a square matrix A, find $Exp(A)$ (easy cases).

  $Exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$

- for a matrix Lie group G and $v \in T_e G$ find 1-parameter subgroup $c(t)$ with $c(0) = e$, $c'(0) = v$.

  $c(t) = Exp(tv)$

- Given a Lie group G and an inner product $< \cdot, \cdot >_e$ on $T_e G$, find the left-invariant Riemannian metric on G.

  $< v, w >_g := < DL_{g^{-1}}(g)v, DL_{g^{-1}}(g)w >_e$

Curvature

- Given (M, g) and a chart $(x_1, \dots, x_n)$, find the components of of the Riemannian tensor R_{ijkl}, R^l_{ijk}.

  $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$
  $R(X,Y,Z,W) := < R(X,Y)Z, W >$
  $R_{ijkl} = R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l})$
  $R^l_{ijk} : \quad R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) \frac{\partial}{\partial x_k} = \sum_l R^l_{ijk} \frac{\partial}{\partial x_l}$
  $R_{ijkl} = \sum_m R^m_{ijk} g_{ml}$
  $R^l_{ijk} = \sum_m R_{ijkm} g^{ml}$

- Given (M, g), $p \in M$ and 2-dimensional plane $\Pi \subset T_p M$, find the sectional curvature $K(\Pi)$.

  $K(\Pi) = K(v_1, v_2) = \frac{< R(v_1,v_2)v_2, v_1 >}{||v_1||^2 ||v_2||^2 - < v_1, v_2 >^2}$ where v_1, v_2 any vectors spanning Π

Jacobi fields

- Given (M, g) and a geodesic $c(t)$ find a basis for Jacobi fields along $c(t)$.

  solve the system of ODE:
  $J''_k + \sum_{k=1}^n R_{kj} J_j = 0$ for all $k = 1, \dots, n$