Riemannian Geometry IV, Term 1: outline

1 Smooth manifolds

"Smooth" means "infinitely differentiable", C^∞.

Definition 1.1. Let M be a set. An n-dimensional smooth atlas on M is a collection of triples $(U_\alpha, V_\alpha, \varphi_\alpha)$, where $\alpha \in I$ for some indexing set I, s.t.

0. $U_\alpha \subseteq M$; $V_\alpha \subseteq \mathbb{R}^n$ is open $\forall \alpha \in I$;
1. $\bigcup_{\alpha \in I} U_\alpha = M$;
2. Each $\varphi_\alpha : U_\alpha \to V_\alpha$ is a bijection;
3. For every $\alpha, \beta \in I$ such that $U_\alpha \cap U_\beta \neq \emptyset$ the composition $\varphi_\beta \circ \varphi_\alpha^{-1}\big|_{\varphi_\alpha(U_\alpha \cap U_\beta)} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is a smooth map for all ordered pairs (α, β), where $\alpha, \beta \in I$.

The number n is called the dimension of M, the maps φ_α are called coordinate charts, the compositions $\varphi_\beta \circ \varphi_\alpha^{-1}$ are called transition maps or changes of coordinates.

Example 1.2. Two atlases on a circle $S^1 \subset \mathbb{R}^2$.

Definition 1.3. Let M have a smooth atlas. A set $A \subseteq M$ is open if for every $\alpha \in I$ the set $\varphi_\alpha(A \cap U_\alpha)$ is open in $\mathbb{R}^n$. This defines a topology on M.

Definition 1.4. M is called Hausdorff if for each $x, y \in M$, $x \neq y$, there exist open sets $A_x \ni x$ and $A_y \ni y$ such that $A_x \cap A_y = \emptyset$.

Example 1.5. An example of a non-Hausdorff space: a line with a double point.

Definition 1.6. M is called a smooth n-dimensional manifold if

1. M has an n-dimensional smooth atlas;
2. M is Hausdorff (in the topology defined by the atlas)
3. M is second-countable (technical condition, we will ignore).

Example 1.7. (a) The boundary of a square in $\mathbb{R}^2$ is a smooth manifold.
(b) Not all sets with a smooth atlas satisfy Hausdorffness condition.

Example 1.8. Examples of smooth manifolds: torus, Klein bottle, 3-torus, real projective space.

Definition 1.9. Let $U \subseteq \mathbb{R}^n$ be open, $m < n$, and let $f : U \to \mathbb{R}^m$ be a smooth map (i.e., all the partial derivatives are smooth). Let $Df(x) = (\frac{\partial f_i}{\partial x_j})$ be the matrix of partial derivatives at $x \in U$ (differential or Jacobi matrix). Then

(a) $x \in \mathbb{R}^n$ is a regular point of f if rk $Df(x) = m$ (i.e., $Df(x)$ has a maximal rank);

(b) $y \in \mathbb{R}^m$ is a regular value of f if the full preimage $f^{-1}(y)$ consists of regular points only.

Theorem 1.10 (Corollary of Implicit Function Theorem). Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}^m$ smooth, $m < n$. If $y \in f(U)$ is a regular value of f then $f^{-1}(y) \subset U \subset \mathbb{R}^n$ is an $(n-m)$-dimensional smooth manifold.

Examples 1.11–1.12. An ellipsoid as a smooth manifold; matrix groups are smooth manifolds.

Definition 1.13. Let M^m and N^n be smooth manifolds of dimensions m and n, with atlases $(U_\alpha, \varphi_\alpha(U_\alpha), \varphi_\alpha)$, $\alpha \in A$ and $(W_\beta, \psi_\beta(W_\beta), \psi_\beta)$, $\beta \in B$. A map $f : M^m \to N^n$ is smooth if it induces smooth maps of open sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, i.e. $\psi_\beta \circ f \circ \varphi_\alpha^{-1}\big|_{\varphi_\alpha(f^{-1}(W_\beta \cap f(U_\alpha)))}$ is smooth for all $\alpha \in A, \beta \in B$.

Remark. A smooth manifold G together with a group operation $G \times G \to G$ is a Lie group is the maps $(g_1, g_2) \to g_1 g_2$ and $g \to g^{-1}$ are smooth.

2 Tangent space

Definition 2.1. A smooth map $f : M \to \mathbb{R}$ is called a smooth function on M.

Definition 2.2. A derivation on the set $C^\infty(M,p)$ of all smooth functions on M defined in a neighborhood of p is a linear map $\delta : C^\infty(M,p) \to \mathbb{R}$, s.t. for all $f, g \in C^\infty(M,p)$ holds $\delta(f \cdot g) = f(p)\delta(g) + \delta(f)g(p)$ (the Leibniz rule).

The set of all derivations is denoted by $\mathcal{D}^\infty(M,p)$. This is a real vector space (exercise).

Definition 2.3. The space $\mathcal{D}^\infty(M,p)$ is called the tangent space of M at p, denoted T_pM. Derivations are tangent vectors.

Definition 2.4. Let $\gamma : (a,b) \to M$ be a smooth curve in M, $t_0 \in (a,b)$, $\gamma(t_0) = p$ and $f \in C^\infty(M,p)$. Define the directional derivative $\gamma'(t_0)(f) \in \mathbb{R}$ of f at p along γ by

$$\gamma'(t_0)(f) = \lim_{s \to 0} \frac{f(\gamma(t_0+s)) - f(\gamma(t_0))}{s} = (f \circ \gamma)'(t_0) = \frac{d}{dt}\bigg|_{t=t_0} (f \circ \gamma)$$

Directional derivatives are derivations (exercise).

Remark. Two curves γ_1 and γ_2 through p may define the same directional derivative.

Notation. Let M^n be a manifold, $\varphi : U \to V \subseteq \mathbb{R}^n$ a chart at $p \in U \subset M$. For $i = 1, \dots, n$ define the curves $\gamma_i(t) = \varphi^{-1}(\varphi(p) + e_i t)$ for small $t > 0$ (here $\{e_i\}$ is a basis of $\mathbb{R}^n$).

Definition 2.5. Define $\frac{\partial}{\partial x_i}\big|_p = \gamma_i'(0)$, i.e.

$$\frac{\partial}{\partial x_i}\bigg|_p (f) = (f \circ \gamma_i)'(0) = \frac{d}{dt}(f \circ \varphi^{-1})(\varphi(p) + te_i)\big|_{t=0} = \frac{\partial}{\partial x_i}(f \circ \varphi^{-1})(\varphi(p)),$$

where $\frac{\partial}{\partial x_i}$ on the right is just a classical partial derivative.

By definition, we have

$$\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \rangle \subseteq \{\text{Directional derivatives}\} \subseteq \mathcal{D}^\infty(M,p)$$

Proposition 2.6. $\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \rangle = \{\text{Directional derivatives}\} = \mathcal{D}^\infty(M,p)$.

Lemma 2.7. Let $\varphi : U \subseteq M \to \mathbb{R}^n$ be a chart, $\varphi(p) = 0$. Let $\tilde\gamma(t) = \left(\sum_{i=1}^n k_i e_i\right) t : \mathbb{R} \to \mathbb{R}^n$ be a line, where $\{e_1, \ldots, e_n\}$ is a basis, $k_i \in \mathbb{R}$. Define $\gamma(t) = \varphi^{-1} \circ \tilde\gamma(t) \in M$. Then $\gamma'(0) = \sum_{i=1}^n k_i \frac{\partial}{\partial x_i}$.

Example. (see Problems class 2) For the group $SL_n(\mathbb{R}) = \{A \in M_n \mid \det A = 1\}$, the tangent space at I is the set of all trace-free matrices: $T_I(SL_n(\mathbb{R})) = \{X \in M_n(\mathbb{R}) \mid \operatorname{tr} X = 0\}$.

Proposition 2.8. (Change of basis for T_pM). Let M^n be a smooth manifold, $\varphi_\alpha : U_\alpha \to V_\alpha$ a chart, $(x_1^\alpha, \ldots, x_n^\alpha)$ the coordinates in V_α. Let $p \in U_\alpha \cap U_\beta$. Then $\left.\frac{\partial}{\partial x_j^\alpha}\right|_p = \sum_{i=1}^n \frac{\partial x_i^\beta}{\partial x_j^\alpha} \frac{\partial}{\partial x_i^\alpha}$, where $\frac{\partial x_i^\beta}{\partial x_j^\alpha} = \frac{\partial(\varphi_\beta^i \circ \varphi_\alpha^{-1})}{\partial x_j^\alpha}(\varphi(p))$, $\varphi_\beta^i = \pi_i \circ \varphi_\beta$.

Definition 2.9. Let M, N be smooth manifolds, let $f : M \to N$ be a smooth map. Define a linear map $Df(p) : T_pM \to T_{f(p)}N$ called the <u>differential</u> of f at p by $Df(p)\gamma'(0) = (f \circ \gamma)'(0)$ for a smooth curve $\gamma \in M$ with $\gamma(0) = p$.

Remark. $Df(p)$ is well defined (i.e. depend only on $\gamma'(0)$ not on the curve γ itself).

Lemma 2.10. If φ is a chart, then
(a) $D\varphi^{-1}(0)$ is linear;
(b) $D\varphi(p) : T_pM \to T_{\varphi(p)}\mathbb{R}^n$ is the identity map taking $\left.\frac{\partial}{\partial x_i}\right|_p$ to $\frac{\partial}{\partial x_i}$;
(c) For $M \xrightarrow{f} N \xrightarrow{g} L$ holds $D(g \circ f)(p) = Dg(f(p)) \circ Df(p)$.

Remark. Hence, $Df(p)$ is a linear map.

Example 2.11. Differential of a map from a disc to a sphere.

Tangent bundle and vector fields

Definition 2.12. Let M be a smooth manifold. A disjoint union $TM = \cup_{p \in M} T_pM$ of tangent spaces to each $p \in M$ is called a <u>tangent bundle</u>.

There is a canonical projection $\Pi : TM \to M$, $\Pi(v) = p$ for every $v \in T_pM$.

Proposition 2.13. The tangent bundle TM has a structure of $2n$-dimensional smooth manifold, s.t. $\Pi : TM \to M$ is a smooth map.

Definition 2.14. A <u>vector field</u> X on a smooth manifold M is a smooth map $X : M \to TM$ such that $\forall p \in M\ X(p) \in T_pM$

The set of all vector fields on M is denoted by $\mathfrak{X}(M)$.

Remark 2.15. (a) $\mathfrak{X}(M)$ has a structure of a vector space.

(b) Vector fields can be multiplied by smooth functions.

(c) Taking a coordinate chart $(U, \varphi = (x_1, \ldots, x_n))$, any vector field X can be written in U as $X(p) = \sum_{i=1}^n f_i(p) \frac{\partial}{\partial x_i} \in T_pM$, where $\{f_i\}$ are some smooth functions on U.

Examples 2.17–2.18. Vector fields on $\mathbb{R}^2$ and 2-sphere.

Remark 2.18. Observe that for $X = \sum a_i(p) \frac{\partial}{\partial x_i} \in \mathfrak{X}(M)$ we have $X(p) \in T_pM$, i.e. $X(p)$ is a directional derivative at $p \in M$. Thus, we can use the vector field to differentiate a function $f \in C^\infty(M)$ by $(Xf)(p) = \sum a_i(p) \left.\frac{\partial f}{\partial x_i}\right|_p$, so that we get another smooth function $Xf \in C^\infty(M)$.

Proposition 2.19. Let $X, Y \in \mathfrak{X}(M)$. Then there exists a unique vector field $Z \in \mathfrak{X}(M)$ such that $Z(f) = X(Y(f)) - Y(X(f))$ for all $f \in C^\infty(M)$.

This vector field $Z = XY - YX$ is denoted by $[X, Y]$ and called the Lie bracket of X and Y.

Proposition 2.20. Properties of Lie bracket:

(a) $[X, Y] = -[Y, X]$;

(b) $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ for $a, b \in \mathbb{R}$;

(c) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (Jacobi identity);

(d) $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$ for $f, g \in C^\infty(M)$.

Definition 2.21. A Lie algebra is a vector space $\mathfrak{g}$ with a binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket which satisfies first three properties from Proposition 2.20.

Proposition 2.20 implies that $\mathfrak{X}(M)$ is a Lie algebra.

Theorem 2.22 (The Hairy Ball Theorem)**.** There is no non-vanishing continuous vector field on an even-dimensional sphere S^{2m}.

3 Riemannian metric

Definition 3.1. Let M be a smooth manifold. A Riemannian metric $g_p(\cdot,\cdot)$ or $\langle \cdot,\cdot \rangle_p$ is a family of real inner products $g_p : T_pM \times T_pM \to \mathbb{R}$ depending smoothly on $p \in M$. A smooth manifold M with a Riemannian metric g is called a Riemannian manifold (M, g).

Examples 3.2–3.3. Euclidean metric on $\mathbb{R}^n$, induced metric on $M \subset \mathbb{R}^n$.

Definition 3.4. Let (M, g) be a Riemannian manifold.
(1) For $v \in T_pM$ define the length of v by $0 \leq \|v\|_g = \sqrt{g_p(v, v)}$.
(2) If $c : [a, b] \to M$ is a smooth curve, then the length of c is $L(c) := \int_a^b ||c'(t)||dt$.

Remark (Reparametrization)**.** Let $\varphi : [c, d] \to [a, b]$ be a strictly monotonic smooth function, $\varphi' \neq 0$, and let $\gamma : [a, b] \to M$ be a smooth curve. Then for $\tilde{\gamma} = \gamma \circ \varphi : [c, d] \to M$ holds $L(\gamma) = L(\tilde{\gamma})$.

Example 3.5. Three models of hyperbolic geometry:

model	notation	M	g
Hyperboloid	$\mathbb{W}^n$	$\{y \in \mathbb{R}^{n+1} \mid q(y, y) = -1, y_{n+1} > 0\}$ where $q(x, y) = \sum_{i=1}^n x_i y_i - x_{n+1} y_{n+1}$	$g_x(v, w) = q(v, w)$
Poincaré ball	$\mathbb{B}^n$	$\{x \in \mathbb{R}^n \mid \|x\|^2 = \sum\limits_{i=1}^n x_i^2 < 1\}$	$g_x(v, w) = \frac{4}{(1-\|x\|^2)^2}\langle v, w\rangle$
Upper half-space	$\mathbb{H}^n$	$\{x \in \mathbb{R}^n \mid x_n > 0\}$	$g_x(v, w) = \frac{1}{x_n^2}\langle v, w\rangle$

Definition 3.6. Given two vector spaces V_1, V_2 with real inner products $(V_i, \langle \cdot,\cdot \rangle_i)$, an isomorphism $T : V_1 \to V_2$ of vector spaces is a linear isometry if $\langle Tv, Tw \rangle_2 = \langle v, w \rangle_1$ for all $v, w \in V_1$.

This is equivalent to preserving the lengths of all vectors (since $\langle v, w \rangle = \frac{1}{2}(\langle v+w, v+w \rangle - \langle v, v \rangle - \langle w, w \rangle)$).

Definition. A bijective map f is a diffeomorphism if both f and f^{-1} are smooth.

Definition 3.7. A diffeomorphism $f : (M, g) \to (N, h)$ of two Riemannian manifolds is an <u>isometry</u> if $Df(p) : T_pM \to T_{f(p)}N$ is a linear isometry for all $p \in M$.

Remark. For a smooth manifold M denote by $Diff(M)$ all diffeomorphisms $M \to M$. Notice that $Diff(M)$ is a group. If (M, g) is a Riemannian manifold, define <u>isometry group</u> $Isom(M, g)$ by $Isom(M, g) := \{f \in Diff(M) \mid f \text{ is an isometry}\}$. $Isom(M, g)$ has a structure of a Lie group.

Example. Some isometry groups:

M	S^1	S^2	$S^2 \setminus \{$ North and South poles$\}$	$S^2 \setminus \{3$ general points$\}$
$Isom(M)$	$O(2)$	$O(3)$	$O(2) \times (\mathbb{Z}/2\mathbb{Z})$	$\{id\}$

Theorem 3.8 (Nash embedding theorem)**.** For any Riemannian manifold (M^m, g) the exists an isometric embedding into $\mathbb{R}^k$ for some $k \in \mathbb{N}$. If M is compact, there exists such $k \leq \frac{m(3m+1)}{2}$, and if M is not compact, there is such $k \leq \frac{m(m+1)(3m+1)}{2}$.

Example 3.9. Isometry between two models of hyperbolic plane.

Definition 3.10. A smooth curve $c : [a, b] \to M$ is <u>arc-length parametrized</u> if $\|c'(t)\| \equiv 1$.

Proposition 3.11 (evident)**.** If a curve $c : [a, b] \to M$ is arc-length parametrized, then $L(c) = b - a$.

Proposition 3.12. Every curve has an arc-length parametrization.

Example 3.13. Length of vertical segments in $\mathbb{H}$. Shortest paths between points on vertical rays.

Definition 3.14. Define a <u>distance</u> $d : M \times M \to [0, \infty)$ on (M, g) by $d(p, q) = \inf_\gamma\{L(\gamma)\}$, where γ is a piecewise smooth curve connecting p and q.

Remark. (M, d) is a metric space.

Definition 3.15. If (M, g) is a Riemannian manifold, then any subset $A \subset M$ is also a metric space with the <u>induced metric</u> $d|_{A\times A} : A \times A \to [0, \infty)$ defined by $d(p, q) = \inf_\gamma\{L(\gamma) \mid \gamma : [a, b] \to A, \gamma(a) = p, \gamma(b) = q\}$, where the length $L(\gamma)$ is computed in M.

Example 3.16. Induced metric on $S^1 \subset \mathbb{R}^2$.

Example 3.17. Punctured Riemann sphere: $\mathbb{R}^n$ with metric $g_x(v, w) = \frac{4}{(1+\|x\|^2)^2}\langle v, w\rangle$.

Definition 3.18. A topological space is <u>compact</u> if for every open cover $\{U_i\}_{i\in I}$ of T there exists a finite subcover.

Example. $\mathbb{R}^n$ is not compact while S^n is compact.

4 Levi-Civita connection and parallel transport

4.1 Levi-Civita connection

Example 4.1. Given a vector field $X = \sum a_i(p)\frac{\partial}{\partial x_i} \in \mathfrak{X}(\mathbb{R}^n)$ and a vector $v \in T_p\mathbb{R}^n$ define the <u>covariant derivative of X in direction v</u> in $\mathbb{R}^n$ by $\nabla_v(X) = \lim\limits_{t\to 0} \frac{X(p+tv)-X(p)}{t} = \sum v(a_i)\frac{\partial}{\partial x_i}\Big|_p \in T_p\mathbb{R}^n$.

Proposition 4.2. The covariant derivative $\nabla_v X$ in $\mathbb{R}^n$ satisfies all the properties (a)–(e) listed below in Definition 4.3 and Theorem 4.4.

Definition 4.3. Let M be a smooth manifold. A map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, $(X, Y) \mapsto \nabla_X Y$ is affine connection if for all $X, Y, Z \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$ holds

(a) $\nabla_X(Y + Z) = \nabla_X(Y) + \nabla_X(Z)$

(b) $\nabla_X(fY) = X(f)Y(p) + f(p)\nabla_X Y$

(c) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$

Theorem 4.4 (Levi-Civita, Fundamental Theorem of Riemannian Geometry)**.** Let (M, g) be a Riemannian manifold. There exists a unique affine connection ∇ on M with the additional properties for all $X, Y, Z \in \mathfrak{X}(M)$:

(d) $v(\langle X, Y \rangle) = \langle \nabla_v X, Y \rangle + \langle X, \nabla_v Y \rangle$ (Riemannian property);

(e) $\nabla_X Y - \nabla_Y X = [X, Y]$ (∇ is torsion-free).

This connection is called Levi-Civita connection of (M, g).

Remark 4.5. Properties of Levi-Civita connection in $\mathbb{R}^n$ and in $M \subset \mathbb{R}^n$ with induced metric.

4.2 Christoffel symbols

Definition 4.6. Let ∇ be the Levi-Civita connection on (M, g), and let $\varphi : U \to V$ be a coordinate chart with coordinates $\varphi = (x_1, \ldots, x_n)$. Since $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}(p) \in T_pM$, there exists a uniquely determined collection of functions $\Gamma_{ij}^k \in C^\infty(U)$ s.t. $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}(p) = \sum_{k=1}^n \Gamma_{ij}^k(p) \frac{\partial}{\partial x_k}(p)$. These functions are called Christoffel symbols of ∇ with respect to the chart φ.

Remark. Christoffel symbols determine ∇ since $\nabla_{\sum\limits_{i=1}^n a_i \frac{\partial}{\partial x_i}} \sum\limits_{j=1}^n b_j \frac{\partial}{\partial x_j} = \sum\limits_{i,j} a_i \frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j} + \sum\limits_{i,j,k} a_i b_j \Gamma_{ij}^k \frac{\partial}{\partial x_k}$.

Proposition 4.7.

$$\Gamma_{ij}^k = \frac{1}{2} \sum_m g^{km}(g_{im,j} + g_{jm,i} - g_{ij,m}),$$

where $g_{ab,c} = \frac{\partial}{\partial x_c} g_{ab}$ and $(g^{ij}) = (g_{ij})^{-1}$, i.e. $\{g^{ij}\}$ are the elements of the matrix inverse to (g_{ij}).

In particular, $\Gamma_{ij}^k = \Gamma_{ji}^k$.

Example 4.8. In $\mathbb{R}^n$, $\Gamma_{ij}^k \equiv 0$ for all i, j, k. Computation of Γ_{ij}^k in $S^2 \subset \mathbb{R}^3$ with induced metric.

4.3 Parallel transport

Definition 4.9. Let $c : (a, b) \to M$ be a smooth curve. A smooth map $X : (a, b) \to TM$ with $X(t) \in T_{c(t)}M$ is called a vector field along c. These fields form a vector space $\mathfrak{X}_c(M)$.

Example 4.10. $c'(t) \in \mathfrak{X}_c(M)$.

Proposition 4.11. Let (M, g) be a Riemannian manifold, let ∇ be the Levi-Civita connection, $c : (a, b) \to M$ be a smooth curve. There exists a unique map $\frac{D}{dt} : \mathfrak{X}_c(M) \to \mathfrak{X}_c(M)$ satisfying

(a) $\frac{D}{dt}(\alpha X + Y) = \alpha \frac{D}{dt} X + \frac{D}{dt} Y$ for any $\alpha \in \mathbb{R}$.

(b) $\frac{D}{dt}(fX) = f'(t)X + f \frac{D}{dt} X$ for every $f \in C^\infty(M)$.

(c) If $\widetilde{X} \in \mathfrak{X}(M)$ is a local extension of X
(i.e. there exists $t_0 \in (a,b)$ and $\varepsilon > 0$ such that $X(t) = \widetilde{X}\big|_{c(t)}$ for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$)
then $(\frac{D}{dt}X)(t_0) = \nabla_{c'(t_0)}\widetilde{X}$.

This map $\frac{D}{dt} : \mathfrak{X}_c(M) \to \mathfrak{X}_c(M)$ is called the covariant derivative along the curve c.

Example 4.12. Covariant derivative in $\mathbb{R}^n$.

Definition 4.13. Let $X \in \mathfrak{X}_c(M)$. If $\frac{D}{dt}X = 0$ then X is said to be parallel along c.

Example 4.14. A vector field X in $\mathbb{R}^n$ is parallel along a curve if and only if X is constant.

Theorem 4.15. Let $c : [a,b] \to M$ be a smooth curve, $v \in T_{c(a)}M$. There exists a unique vector field $X \in \mathfrak{X}_c(M)$ parallel along c with $X(a) = v$.

Corollary 4.16. Parallel vector fields form a vector space of dimension n (where n is the dimension of (M,g)).

Definition 4.17. Let $c : [a,b] \to M$ be a smooth curve. A linear map $P_c : T_{c(a)}M \to T_{c(b)}M$ defined by $P_c(v) = X(b)$, where $X \in \mathfrak{X}_c(M)$ is parallel along c with $X(a) = v$, is called a parallel transport along c.

Remark. The parallel transport P_c depends on the curve c (not only on its endpoints).

Proposition 4.18. The parallel transport $P_c : T_{c(a)}M \to T_{c(b)}M$ is a linear isometry between $T_{c(a)}M$ and $T_{c(b)}M$, i.e. $g_{c(a)}(v,w) = g_{c(b)}(P_c v, P_c w)$.

5 Geodesics

5.1 Geodesics as solutions of ODE's

Definition 5.1. Given (M,g), a curve $c : [a,b] \to M$ is a geodesic if $\frac{D}{dt}c'(t) = 0$ for all $t \in [a,b]$ (i.e., $c'(t) \in \mathfrak{X}_c(M)$ is parallel along c).

Lemma 5.2. If c is a geodesic then c is parametrized proportionally to the arc length.

Theorem 5.3. Given a Riemannian manifold (M,g), $p \in M$, $v \in T_pM$, there exists $\varepsilon > 0$ and a unique geodesic $c : (-\varepsilon, \varepsilon) \to M$ such that $c(0) = p$, $c'(0) = v$.

Examples 5.4–5.5 Geodesics in Euclidean space, on a sphere, and in the upper half-plane model $\mathbb{H}^2$.

5.2 Geodesics as distance-minimizing curves. First variation formula of the length

Definition 5.6. Let $c : [a,b] \to M$ be a smooth curve. A smooth map $F : (-\varepsilon,\varepsilon) \times [a,b] \to M$ is a (smooth) variation of c if $F(0,t) = c(t)$. Variation is proper if $F(s,a) = c(a)$ and $F(s,b) = c(b)$ for all $s \in (-\varepsilon, \varepsilon)$.

Variation can be considered as a family of the curves $F_s(t) = F(s,t)$.

Definition 5.7. A variational vector field $X \in \mathfrak{X}_c(M)$ of a variation F is defined by $X(t) = \frac{\partial F}{\partial s}(0,t)$.

Definition 5.8. The length $l : (-\varepsilon,\varepsilon) \to [0,\infty)$ and energy $E : (-\varepsilon,\varepsilon) \to [0,\infty)$ of a variation F are defined by

$$l(s) = \int_a^b \|\frac{\partial F}{\partial t}(s,t)\|\,\mathrm{d}t, \qquad E(s) = \frac{1}{2}\int_a^b \|\frac{\partial F}{\partial t}(s,t)\|^2\,\mathrm{d}t$$

Remark. $l(s)$ is the length of the curve $F_s(t)$.

Theorem 5.9. A smooth curve c is a geodesic if and only if c is parametrized proportionally to the arc length and $l'(0) = 0$ for every proper variation of c.

Corollary 5.10. Let $c : [a, b] \to M$ be the shortest curve from $c(a)$ to $c(b)$, and c is parametrized proportionally to the arc length. Then c is geodesic.

Remark. The converse is false (e.g., on the sphere).

Lemma 5.11 (Symmetry Lemma)**.** Let $W \subset \mathbb{R}^2$ be an open set and $F : W \to M$, $(s,t) \mapsto F(s,t)$, be a smooth map. Let $\frac{D}{dt}$ be the covariant derivative along $F_s(t)$ and $\frac{D}{ds}$ be the covariant derivative along $F_t(s)$. Then $\frac{D}{dt}\frac{\partial F}{\partial s} = \frac{D}{ds}\frac{\partial F}{\partial t}$.

Theorem 5.12 (First variation formula of length)**.** Let $F : (-\varepsilon, \varepsilon) \times [a, b] \to M$ be a variation of a smooth curve $c(t)$, $c'(t) \neq 0$. Let $X(t)$ be its variational vector field and $l : (-\varepsilon, \varepsilon) \to [0, \infty)$ its length. Then

$$l'(0) = \int_a^b \frac{1}{\|c'(t)\|} \frac{d}{dt} \langle X(t), c'(t) \rangle \, \mathrm{d}t - \int_a^b \frac{1}{\|c'(t)\|} \langle X(t), \frac{D}{dt} c'(t) \rangle \, \mathrm{d}t$$

Corollary 5.13. (a) If $c(t)$ is parametrized proportionally to the arc length, $\|c'(t)\| \equiv c$, then $l'(0) = \frac{1}{c}\langle X(b), c'(b)\rangle - \frac{1}{c}\langle X(a), c'(a)\rangle - \frac{1}{c}\int_a^b \langle X(t), \frac{D}{dt}c'(t)\rangle \, \mathrm{d}t$;

(b) if $c(t)$ is geodesic, then $l'(0) = \frac{1}{c}\langle X(b), c'(b)\rangle - \frac{1}{c}\langle X(a), c'(a)\rangle$;

(c) if F is proper <u>and</u> c is parametrized proportionally to the arc length, then $l'(0) = -\frac{1}{c}\int_a^b \langle X(t), \frac{D}{dt}c'(t)\rangle \, \mathrm{d}t$;

(d) if F is proper <u>and</u> c is geodesic, then $l'(0) = 0$.

Lemma 5.14. Any vector field X along $c(t)$ with $X(a) = X(b) = 0$ is a variational vector field for some proper variation F.

5.3 Exponential map and Gauss Lemma

Let $p \in M$, $v \in T_pM$. Denote by $c_v(t)$ the unique maximal geodesic (i.e., the domain is maximal) with $c_v(0) = p$, $c_v'(0) = v$.

Definition 5.15. If $c_v(1)$ exists, define $\exp_p : T_pM \to M$ by $\exp_p(v) = c_v(1)$, the <u>exponential map</u> at p.

Example 5.16. Exponential map on the sphere S^2: length of c_v from p to $c_v(1)$ equals $\|v\|$.

Notation. $B_r(0_p) = \{v \in T_pM \mid \|v\| < r\} \subset T_pM$ is a ball of radius r centered at 0_p.

Proposition 5.17. (without proof)
For any $p \in (M, g)$ there exists $r > 0$ such that $\exp_p : B_r(0_p) \to \exp_p(B_r(0_p))$ is a diffeomorphism.

Example. On S^2 the set $\exp_p(B_{\pi/2}(0_p))$ is a hemisphere, so that every geodesic starting from p is orthogonal to the boundary of this set.

Theorem 5.18 (Gauss Lemma)**.** Let (M, g) be a Riemannian manifold, $p \in M$, and let $\varepsilon > 0$ be such that $\exp_p : B_\varepsilon(0_p) \to \exp_p(B_\varepsilon(0_p))$ is a diffeomorphism. Define $A_\delta = \{\exp_p(w) \mid \|w\| = \delta\}$ for every $0 < \delta < \varepsilon$. Then every <u>radial</u> geodesic $c : t \mapsto \exp_p(tv)$, $t \geq 0$, is orthogonal to A_δ.

Remark 5.19. The curve $c_v(t) = \exp_p(tv)$ is indeed geodesic; every geodesic γ through p can be written as $\gamma(t) = \exp_p(tw)$ for appropriate $w \in T_pM$.

Definition. Denote $B_\varepsilon(p) = \exp_p(B_\varepsilon(0_p)) \subset M$, a geodesic ball.

Lemma 5.20. Let (M, g) be a Riemannian manifold and $p \in M$. Let $\varepsilon > 0$ be small enough such that $\exp_p : B_\varepsilon(0_p) \to B_\varepsilon(p) \subset M$ is a diffeomorphism. Let $\gamma : [0, 1] \to B_\varepsilon(p) \setminus \{p\}$ be any curve. Then there exists a curve $v : [0, 1] \to T_pM$, $\|v(s)\| = 1$ for all $s \in [0, 1]$, and a positive function $r : [0, 1] \to \mathbb{R}_+$, such that $\gamma(s) = \exp_p(r(s)v(s))$.

Lemma 5.21. Let $r : [0, 1] \to \mathbb{R}_+$, $v : [0, 1] \to S_pM = \{w \in T_pM \mid \|w\| = 1\}$. Define $\gamma : [0, 1] \to B_\varepsilon(p) \setminus \{p\}$ by $\gamma(s) = \exp_p(r(s)v(s))$. Then the length $l(\gamma) \geq |r(1) - r(0)|$, and the equality holds if and only if γ is a reparametrization of a radial geodesic (i.e. $v(s) \equiv \|v(0)\|$ and $r(s)$ is a strictly increasing or decreasing function).

Corollary 5.22. Given a point $p \in M$, there exists $\varepsilon > 0$ such that for any $q \in B_\varepsilon(p)$ there exists a curve $c(t)$ connecting p and q and satisfying $l(c) = d(p, q)$. (This curve is a radial geodesic).

Remark. According to Corollary 5.22, there is $\varepsilon > 0$ such that $B_\varepsilon(p)$ coincides with ε-ball at p, i.e. with $\{q \in M \mid d(p, q) < \varepsilon\}$.

Proposition 5.23. (without proof)
Let $p \in M$. Then there is an open neighborhood U of p and $\varepsilon > 0$ such that $\forall\, q \in U \;\; \exp_q : B_\varepsilon(0_q) \to B_\varepsilon(q)$ is a diffeomorphism.

5.4 Hopf-Rinow Theorem

Definition 5.24. A geodesic $c : [a, b] \to M$ is minimal if $l(c) = d(c(a), c(b))$. A geodesic $c : \mathbb{R} \to M$ is minimal if its restriction $c|_{[a,b]}$ is minimal for each segment $[a, b] \subset \mathbb{R}$.

Example. No minimal geodesics in S^2, all geodesics in $\mathbb{E}^2$ are minimal.

Definition 5.25. A Riemannian manifold (M, g) is geodesically complete if every geodesic $c : [a, b] \to M$ can be extended to a geodesic $\tilde{c} : \mathbb{R} \to M$ (i.e. can be extended infinitely in both directions). Equivalently, $\exp_p$ is defined on the whole T_pM for all $p \in M$.

Theorem 5.26 (Hopf-Rinow)**.** Let (M, g) be a connected Riemannian manifold with distance function d. Then the following are equivalent:

(a) (M, g) is complete (i.e. every Cauchy sequence converges);

(b) every closed and bounded subset is compact;

(c) (M, g) is geodesically complete.

Moreover, every of the conditions above implies

(d) for every $p, q \in M$ there exists a minimal geodesic connecting p and q.

Remark. A geodesic in (d) may not be unique. Further, (d) does not imply (c).

Remark. Theorem 5.26 uses the following notions defined in a metric space:

- $\{x_i\}$, $x_i \in M$, is a Cauchy sequence if $\forall \varepsilon > 0\, \exists N\, \forall\, m, n > N \;\; d(x_m, x_n) < \varepsilon$;

- a set $A \subset M$ is bounded if $A \subset B_r(p)$ for some $r > 0$, $p \in M$;
- a set $A \subset M$ is closed if $\{x_n \in A, x_n \to x\} \Rightarrow x \in A$;
- a set $A \subset M$ is compact if each open cover has a finite subcover;
- a set $A \subset M$ is sequentially compact if each sequence has a converging subsequence.

Some properties:
1. A compact set is sequentially compact, bounded, closed.
2. A compact metric space is complete.
3. In a complete metric space, a sequentially compact set is compact.

Integration on Riemannian manifolds

Definition. A support of a function $f : M \to \mathbb{R}$ is the set $supp(f) := \{x \in M \mid f(x) \neq 0\}$.

Definition. Let (M, g) be a Riemannian manifold (M, g) and a let $f : M \to \mathbb{R}$ be a function. Let $\varphi : U \to V$, $U \subset M$, $V \subset \mathbb{R}^n$ be a chart, $\varphi = (x_1, \dots, x_n)$. Assume that $supp(f) \subset U$. Then

$$\int\limits_M f = \int\limits_M f d\ Vol = \int\limits_U f d\ Vol = \int\limits_V f \circ \varphi^{-1}(x)\sqrt{det(g_{ij} \circ (\varphi^{-1}(x))}dx,$$

where $g_{ij}(p) = \langle \frac{\partial}{\partial x_i}\big|_p, \frac{\partial}{\partial x_j}\big|_p \rangle$ is the metric g written in the chart φ.

Remark. The result does not depend on the choice of the chart.

Definition. A volume of a (good) subset $A \subset U \subset M$ is defined by $VolA = \int\limits_M 1_A d\ Vol$, where 1_A is a characteristic function of A: $1_A(p) = 1$ for all $p \in A$ and $1_A(p) = 0$ otherwise. In other words,

$$Vol(A) = \int\limits_A d\ Vol = \int\limits_{\varphi(A)} \sqrt{det(g_{ij} \circ \varphi^{-1}(x))}dx.$$

Example. The area of a hyperbolic triangle with all three vertices on the boundary is π (computation in the upper half-plane model).

Remark. If $supp(f)$ does not lie in one chart, one uses the technique of partition of unity which we don't study in this course.