Finite mutation type in presence of coefficients

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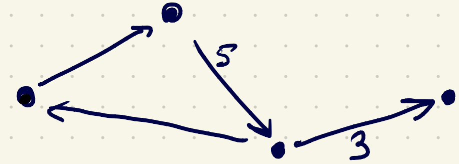
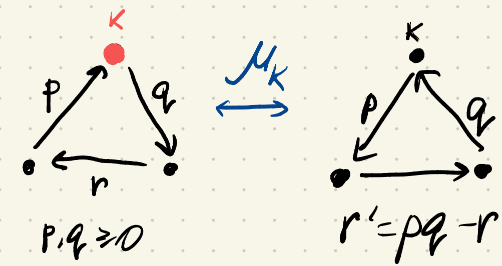
CARW03: Interdisciplinary applications of cluster algebras

- Consider a quiver Q



  (no loops, no 2-cycles)

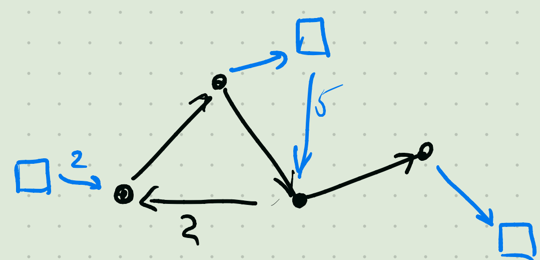
  with usual mutation rule:



- When is it mutation-finite?

- Answer is known when Q has no frozen vertices
  (frozen = unmutable)

- **Question:** what if Q has frozen vertices?



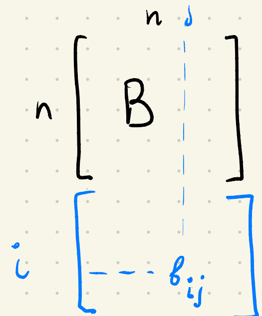
???

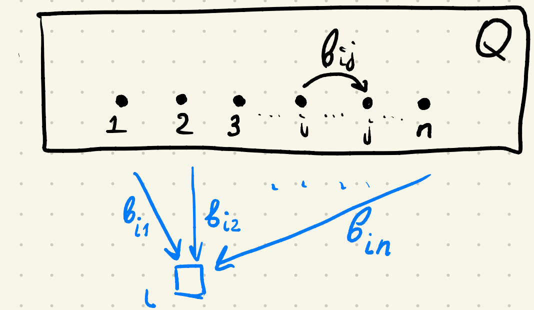
- **Rules**: as in [FZ4]
  - no mutations in frozen
  - no arrows between frozen

$$b'_{ij} = \begin{cases} -b_{ij} & i=k \text{ or } j=k \\ b_{ij} + \operatorname{sgn}(b_{ik})\,[b_{ik} b_{kj}]_+ & \text{otherwise} \end{cases}$$

$$[x]_+ = \max\{x, 0\}$$

- In matrix notation:





- Two observations:
  1) If the ice quiver is mutation-finite, then mutable part Q is mutation-finite too.
  2) Only need to study the case of one frozen vertex (as frozen vertices do not interact)

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  1) If the ice quiver is mutation-finite, then mutable part Q is mutation-finite too.
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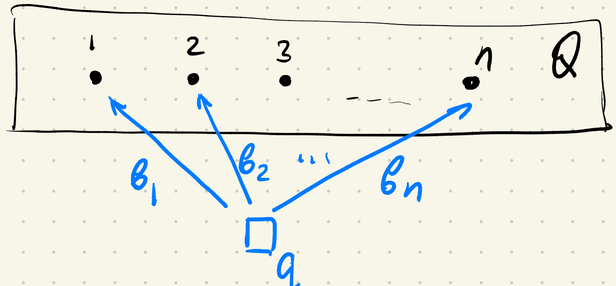
- Notation:

$\underline{b} = (b_1, b_2, \dots, b_n)$



- The vector $(b_1, b_2, \dots, b_n)$ will be called admissible if the ice quiver $\langle Q, q\rangle$ is mutation finite

- Aim: describe all admissible vectors for every $\overset{\text{mut.- finite}}{Q}$

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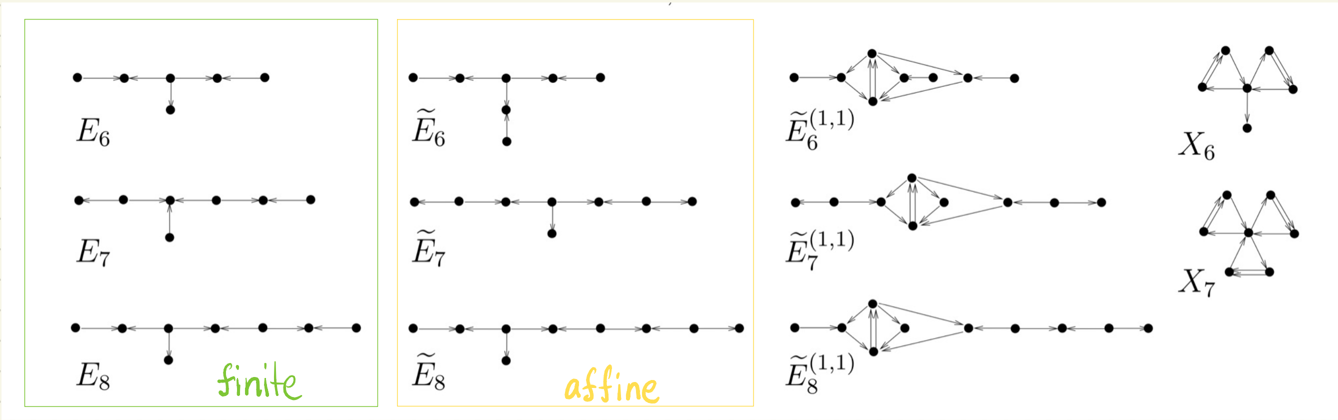
  mut-finite

- [FZ4]: If Q is of finite type, then every b is admissible

- [FZ4]: this is a characteristic property of finite type.

- [Seven]: If Q with principle coefficients is mutation-finite then Q is of finite type

- **Aim:** describe all admissible vectors for every Q (mut - finite)
- **FZ4:** If Q is of finite type, then <u>every</u> <u>b</u> is admissible
- **Classification** [F, Sh, T]: Connected Q is mut-fin
  $\iff$ (1) either Q is from a triang surface,
  (2) or Q is of rank 2; (3) or Q is mut-equiv. to one of:



surfaces

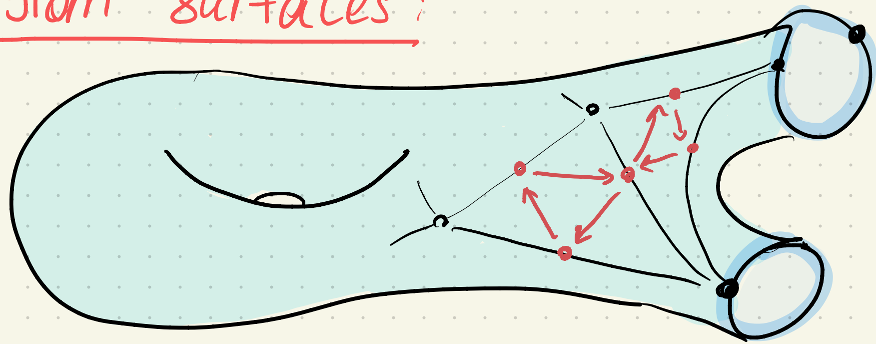
finite
all

affine

otherwise

1. Quivers from surfaces:

[Fomin, Shapiro, Thurston]

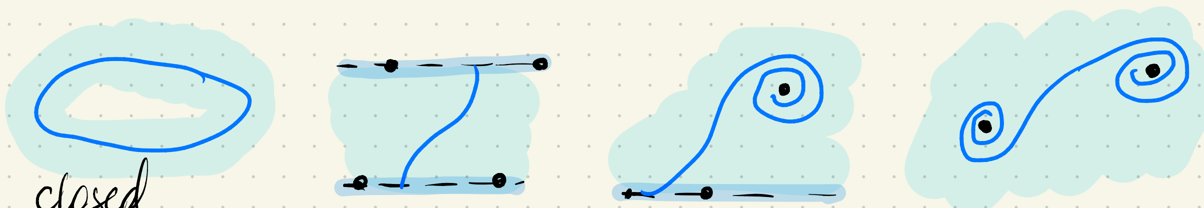


$\rightsquigarrow Q$

- mutation μ_k
  $\updownarrow$
  flip of T

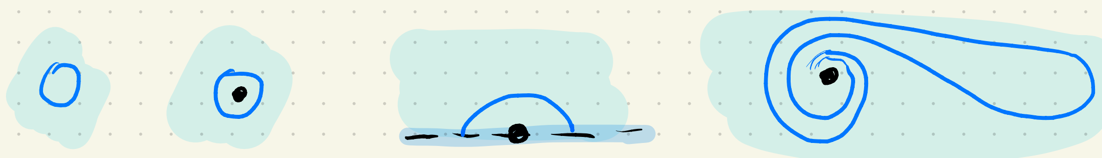
- **frozen vertices** are represented by **laminations**: [Fomin, Thurston]
  i.e. non-intersecting set of curves:

allowed:



closed

not allowed:



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1. Quivers from surfaces:

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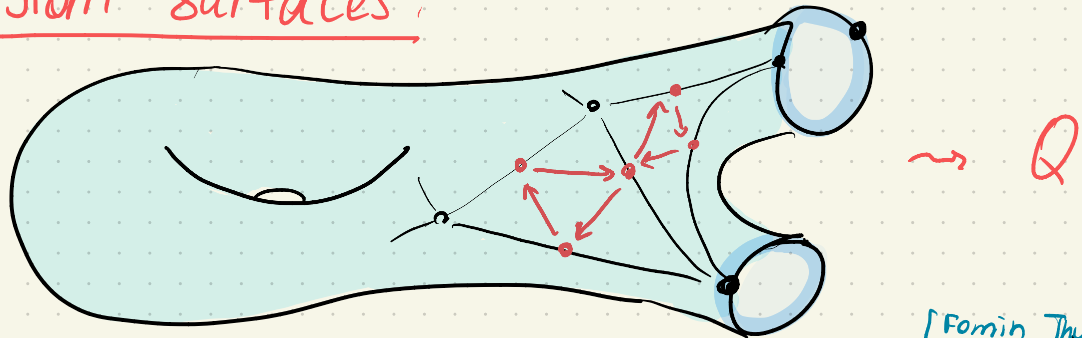
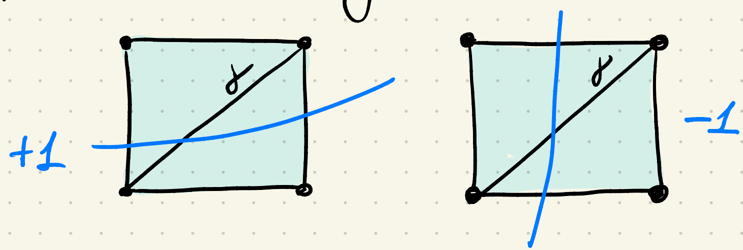
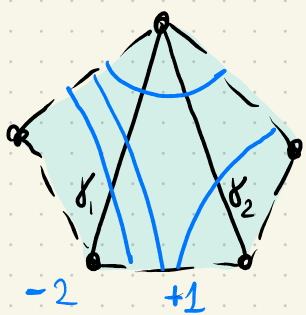


- frozen vertices are represented by laminations: [Fomin, Thurston]
- Shear coordinates:



Ex



$b_{\gamma_1}(L) = -2$

$b_{\gamma_2}(L) = 1$ $\rightsquigarrow$ $b = (2, -1)$

surfaces

finite

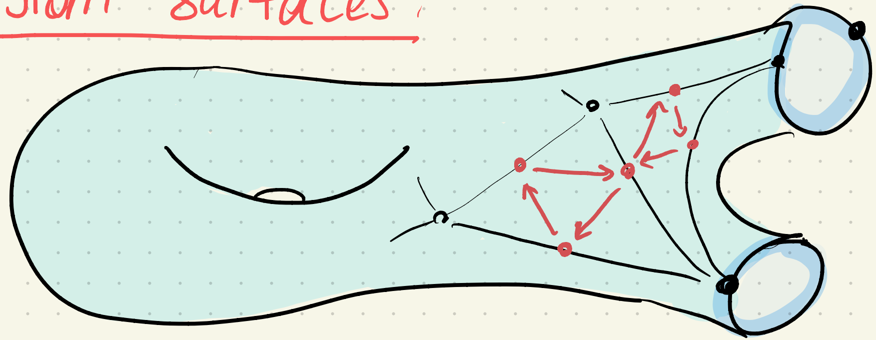
all

affine

otherwise

1. Quivers from surfaces:

[Fomin, Shapiro, Thurston]

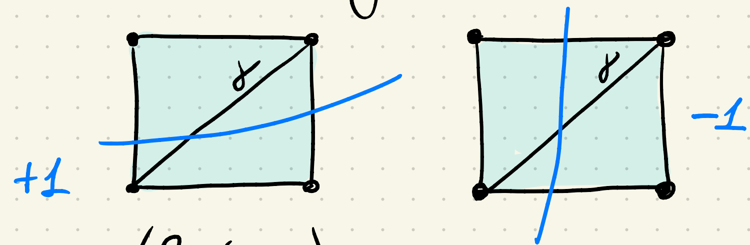


$\rightsquigarrow Q$

- mutation μ_k
  $\updownarrow$
  flip of T

[Fomin, Thurston]

- frozen vertices are represented by **laminations**:
- Shear coordinates:



- [Fomin, Thurston] ① The map $L \to (b_i(T,L))$ is a bijection between laminations and $\mathbb{Z}^n$

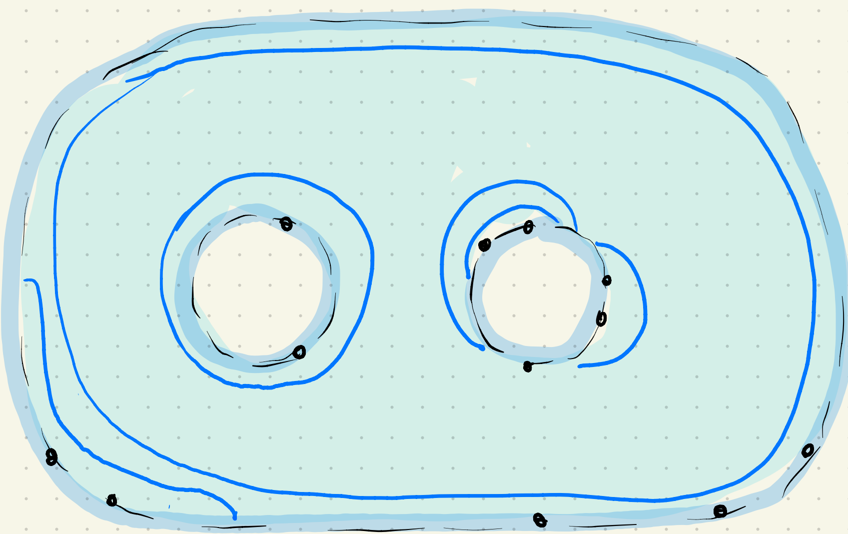
② The vector of shear coordinates is mutated in the same way as coefficient vector $(b_1, \dots, b_n)$

surfaces	
finite	all
affine	
otherwise	

1. Quivers from surfaces:

<u>Thm</u> Let Q be from triangulated surface S.
Then

$$\left\{\begin{matrix}\text{admissible vectors}\\ \text{for } Q\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{peripheral}\\ \text{laminations for } S\end{matrix}\right\}$$



" ones that could be isotopically deformed to boundary

surfaces peripheral laminations
finite all
affine
otherwise

1. Quivers from surfaces:

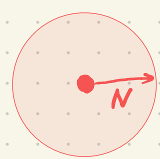
Thm Let Q be from triangulated surface S.
Then

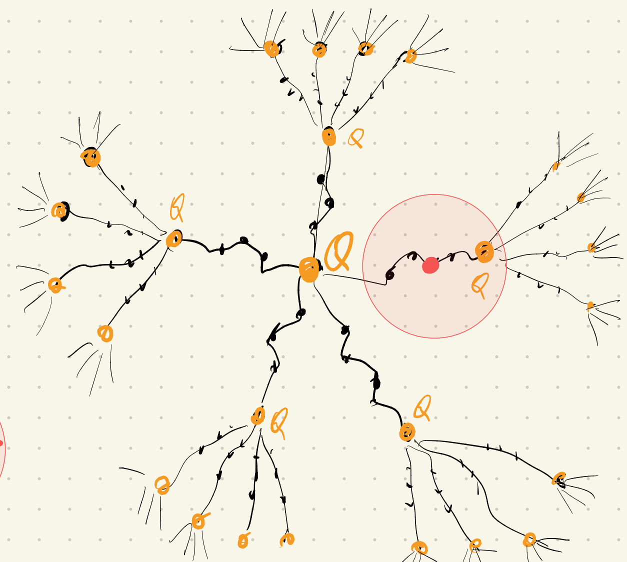
$$\left\{\begin{array}{l}\text{admissible vectors}\\ \text{for } Q\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{peripheral}\\ \text{laminations for } S\end{array}\right\}$$

"Pf" peripheral lam = ?

= ones preserved by all Dehn twists

= preserved by modular group





surfaces

peripheral laminations

finite

all

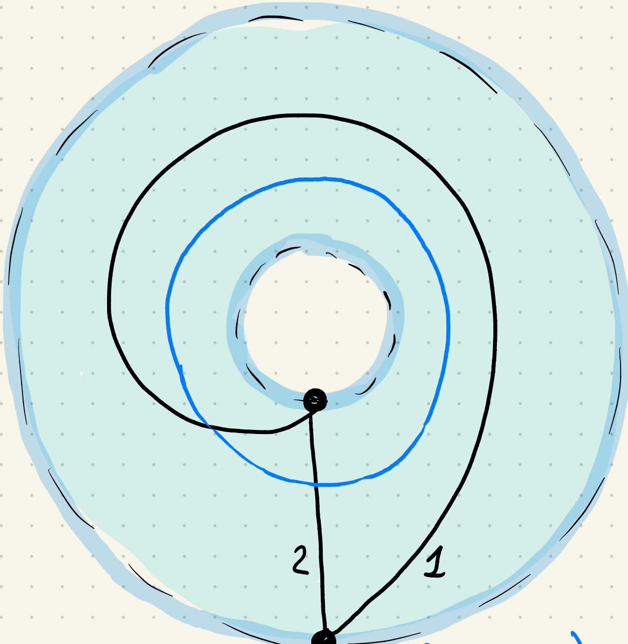
affine

otherwise

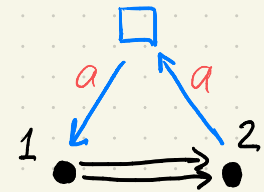
1. Quivers from surfaces:

Ex $\tilde{A}_1$

$$1 \bullet \rightrightarrows \bullet 2$$



$(+1,-1) \rightsquigarrow \underline{b} = (-1,1)$

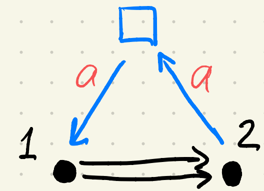


"Annulus property"

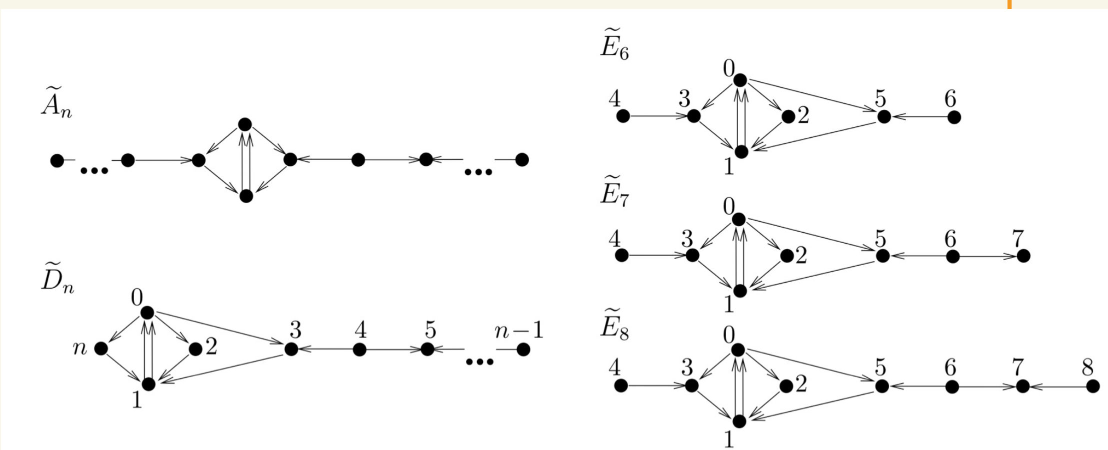
surfaces peripheral laminations
finite all
affine
otherwise

2. Affine type

<u>Thm</u> Let Q be affine quiver containing a double arrow. Then $\underline{b}$ is admissible $\Leftrightarrow$ $\underline{b}$ satisfies the annulus property



"Annulus property"



<u>Note</u>: Every aff. mut. class contains a quiver with a double arrow.

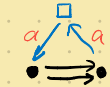
surfaces

peripheral laminations

finite

all

affine



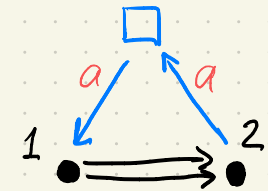
annulus prop.

otherwise

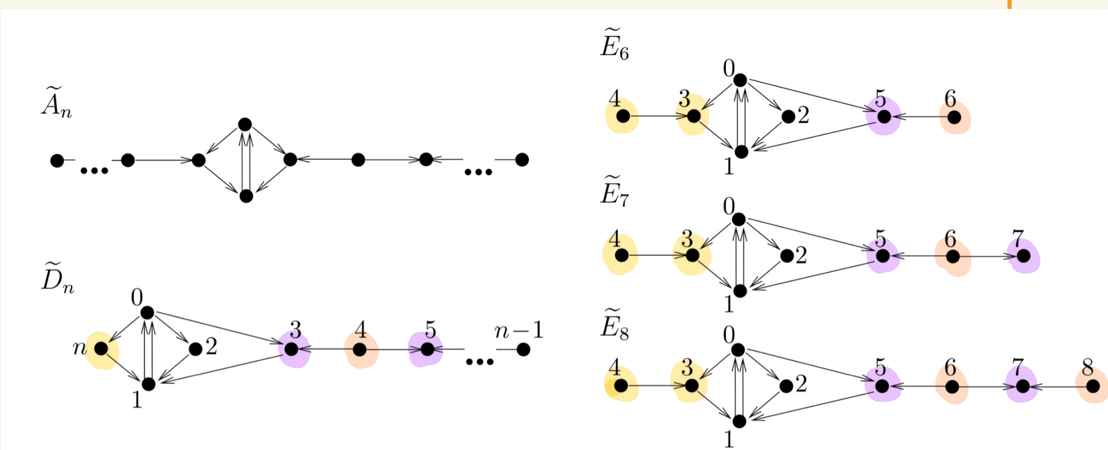
2. Affine type

Idea of pf: • use quivers below

• apply result from [Kaufman, Greenberg] computing cluster modular group



"Annulus property"

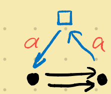


Generators:

$$\mu^{(1)} = \mu_2 \circ \mu_1 \circ \mu_0$$

$$\mu^{(2)} = \mu_4 \circ \mu_3 \circ \mu_1 \circ \mu_0$$

$$\mu^{(3)} = \mu \circ \mu \circ \mu_1 \circ \mu_0$$

surfaces	peripheral laminations
finite	all
affine	 annulus prop
otherwise	

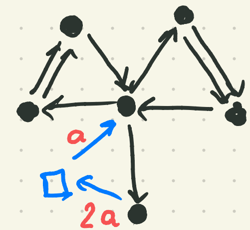
3. Everything else $\left(E_6^{(1,1)}, E_7^{(1,1)}, E_8^{(1,1)}, X_6, X_7\right)$

Thm There are NO admissible vectors for

$$Q = E_6^{(1,1)}, E_7^{(1,1)}, E_8^{(1,1)}, X_7.$$

For X_6,

all admissible vectors are as follows :

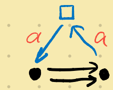


Why? We build a sequence of mutations μ s.t if annulus property is satisfied for Q, then it is not satisfied for $\mu(Q)$.

surfaces
peripheral laminations

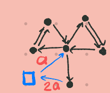
finite
all

affine



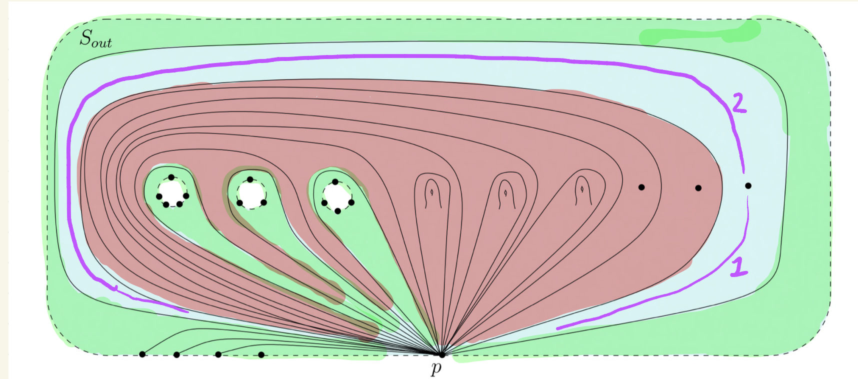
annulus prop

otherwise
None
except for X_6

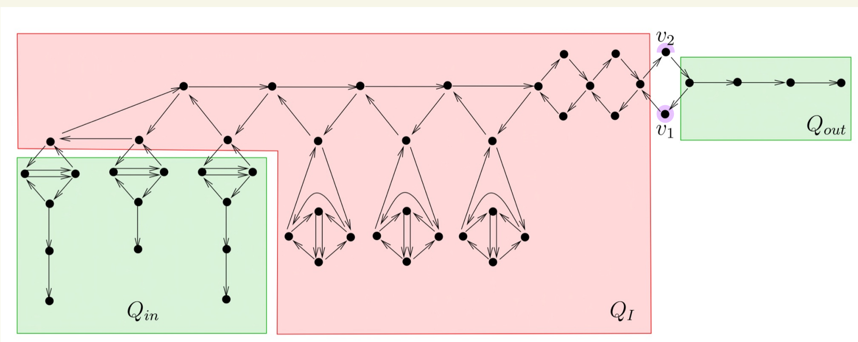


4. More explicit result for surfaces

- Take "nested" triangulation



holes handles punctures outer bdy



Thm b admissible $\Leftrightarrow$

1) $b_i = 0$ $\forall i \in I$

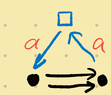
2) annulus property is satisfied

3) $b_1 = -b_2 \leq 0$

surfaces peripheral laminations

finite: all

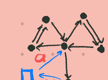
affine



annulus prop

otherwise: None

except for X_6



5. Skew-symmetrizable case

everything extends with similar answers, but:

- orbifolds in place of surfaces;
- diagrams in place of quivers;
- annulus property with weights;

[for diagram $\overset{1}{\bullet} \xrightarrow{4} \overset{2}{\bullet}$ need $d_1 b_1^2 = d_2 b_2^2$, $b_1 \le 0 \le b_2$] i.e. $-b_1 = b_2 \ge 0$ if $(d_1, d_2) = (1,1)$; $-b_1 = 2b_2 \ge 0$ if $(d_1, d_2) = (1,4)$

- unfoldings for avoiding computations [both in affine and extended aff. case]

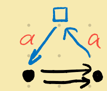
surfaces

peripheral laminations

finite

all

affine

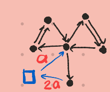


annulus prop

otherwise

None

except for X_6



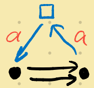
surfaces

peripheral laminations

finite

all

affine



annulus prop

otherwise

None

except for X_6