Cluster algebras, quiver mutations and triangulated surfaces

Anna Felikson

Durham University

joint work with **Michael Shapiro** and **Pavel Tumarkin**

IMA Early Career Mathematicians Conference
April 21, 2018

Cluster algebras

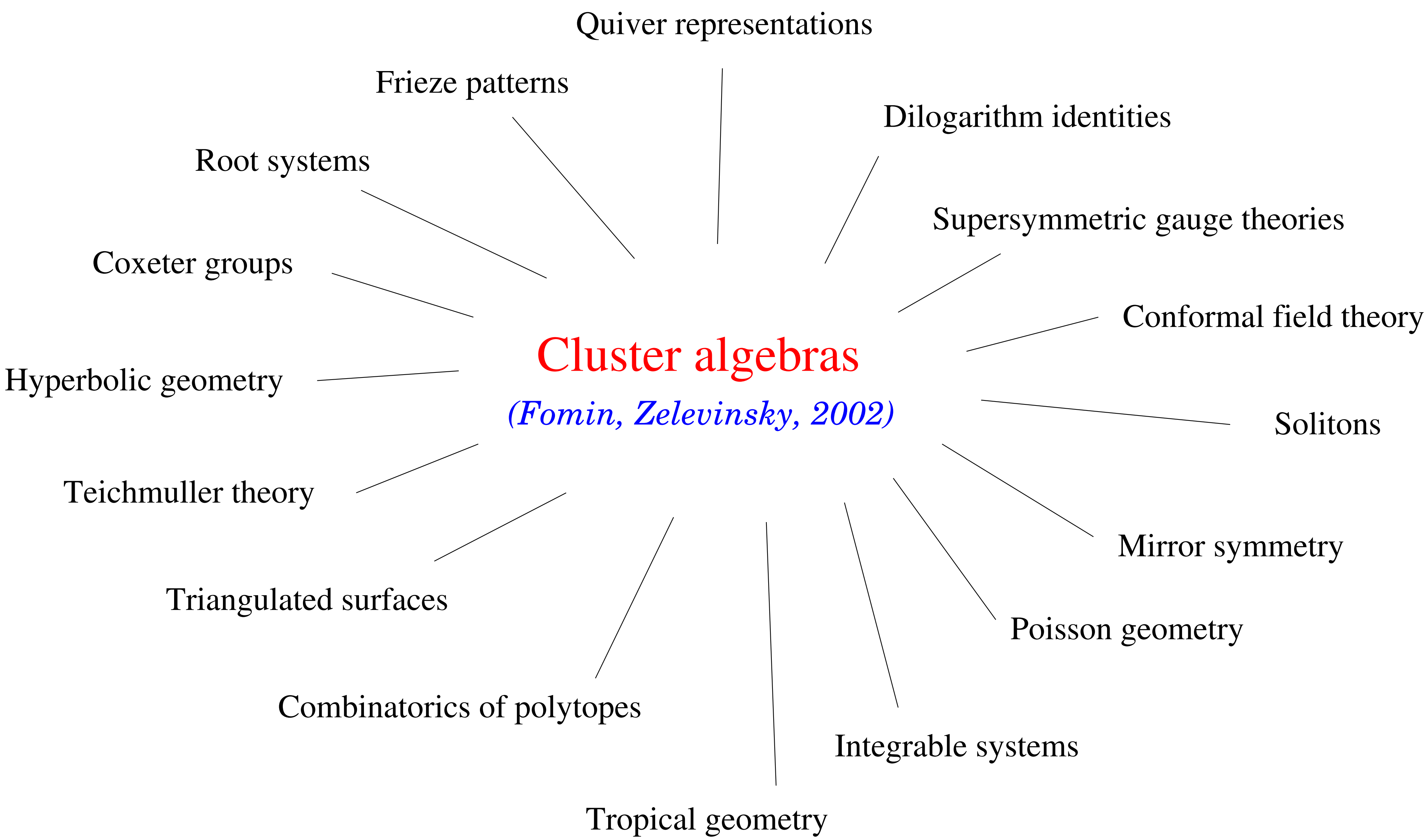
(Fomin, Zelevinsky, 2002)

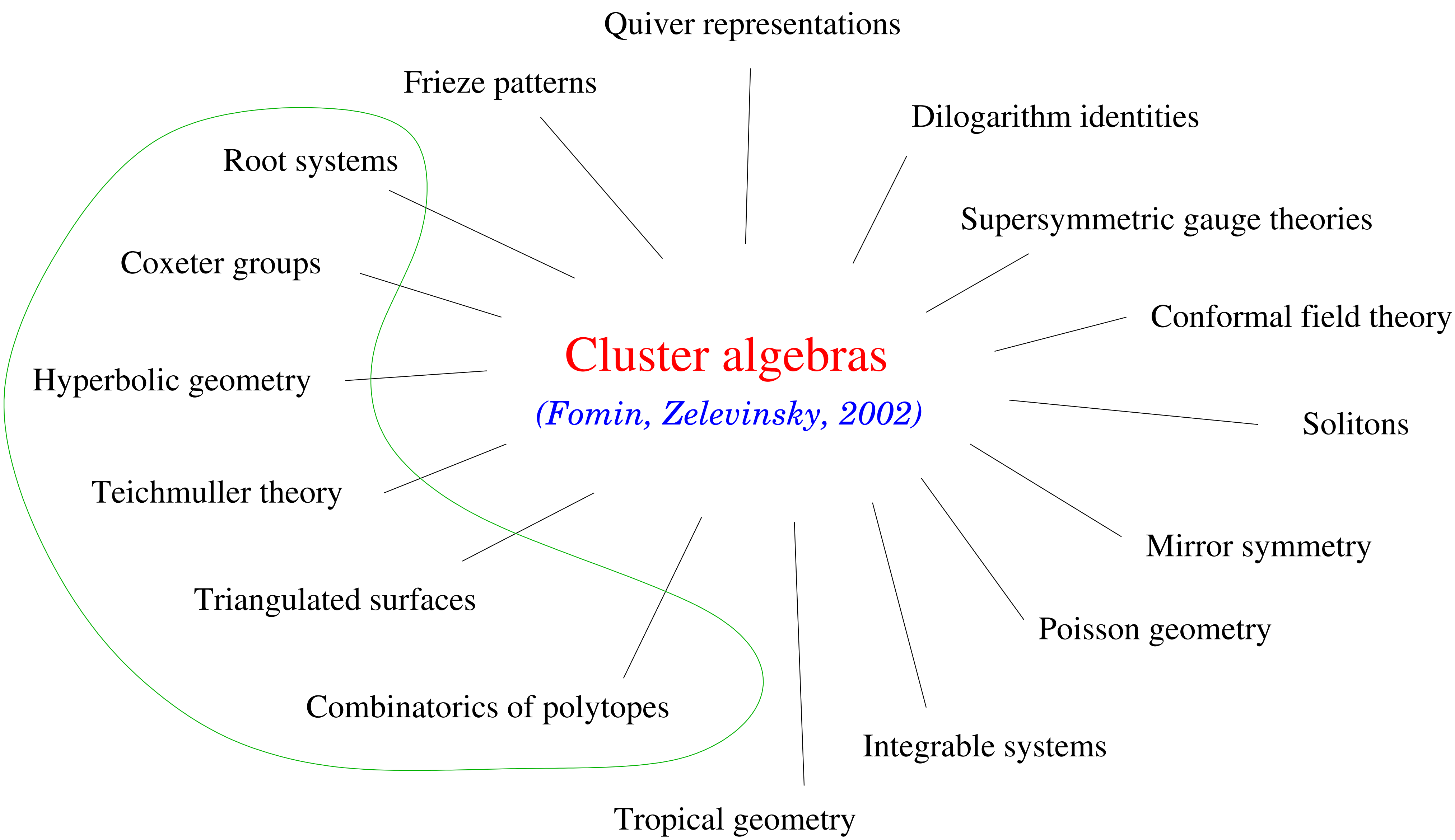
Andrei Zelevinsky

Cluster algebras

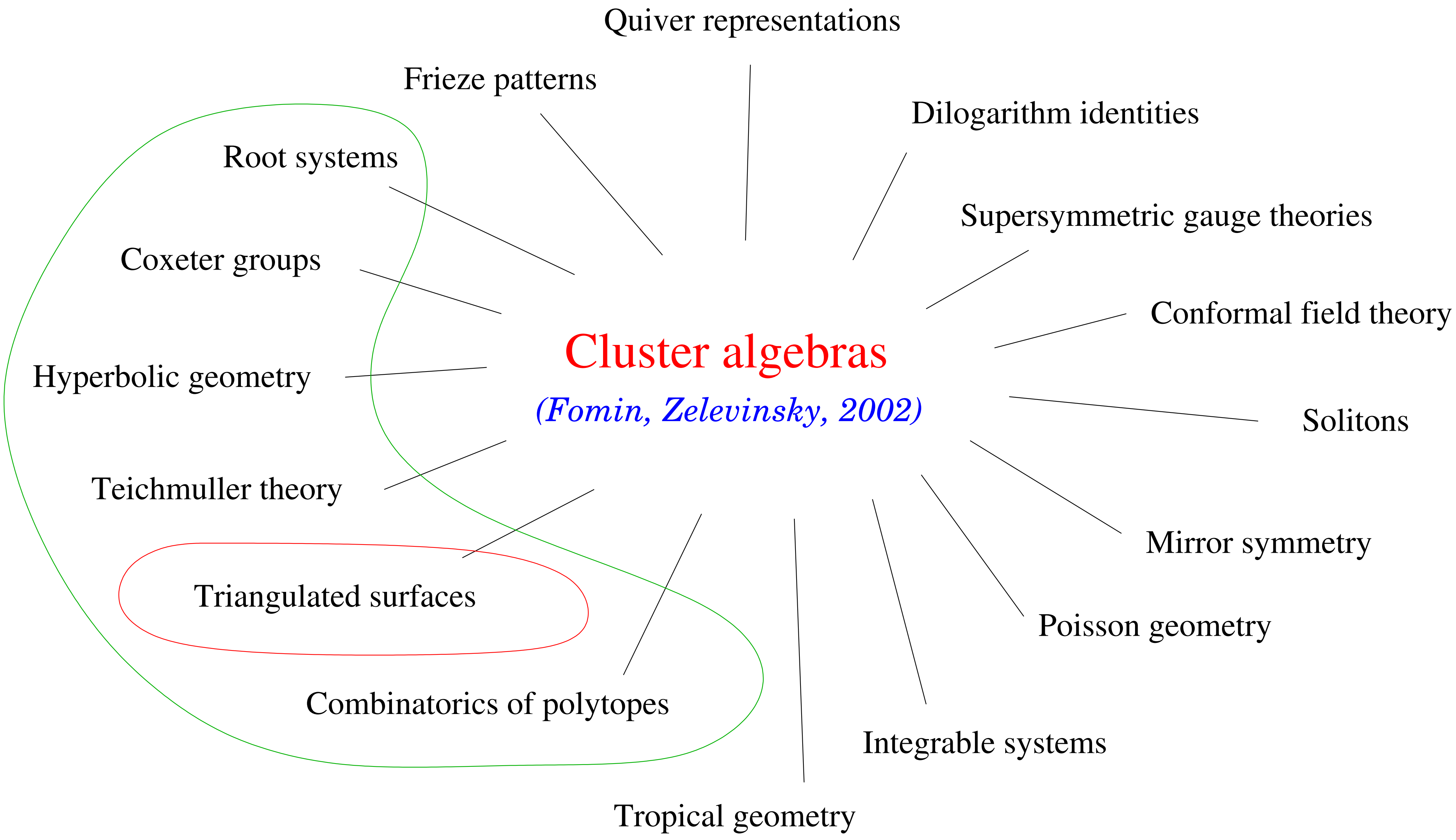
(Fomin, Zelevinsky, 2002)

Sergey Fomin


Cluster algebras
(Fomin, Zelevinsky, 2002)
Quiver representations
Frieze patterns
Dilogarithm identities
Root systems
Supersymmetric gauge theories
Coxeter groups
Conformal field theory
Hyperbolic geometry
Solitons
Teichmuller theory
Mirror symmetry
Triangulated surfaces
Poisson geometry
Combinatorics of polytopes
Integrable systems
Tropical geometry



Cluster algebras
(Fomin, Zelevinsky, 2002)
Quiver representations
Frieze patterns
Dilogarithm identities
Root systems
Supersymmetric gauge theories
Coxeter groups
Conformal field theory
Hyperbolic geometry
Solitons
Teichmuller theory
Mirror symmetry
Triangulated surfaces
Poisson geometry
Combinatorics of polytopes
Integrable systems
Tropical geometry


Quiver representations
Frieze patterns
Dilogarithm identities
Root systems
Supersymmetric gauge theories
Coxeter groups
Conformal field theory
Cluster algebras
(Fomin, Zelevinsky, 2002)
Hyperbolic geometry
Solitons
Teichmuller theory
Mirror symmetry
Triangulated surfaces
Poisson geometry
Combinatorics of polytopes
Integrable systems
Tropical geometry

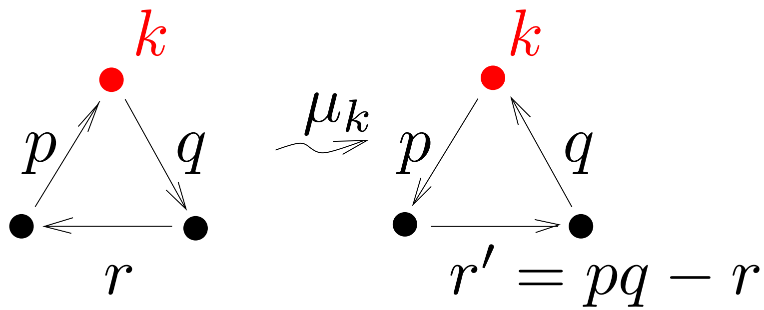
1. Quiver mutation

- Quiver is a directed graph without loops and 2-cycles.

1. Quiver mutation

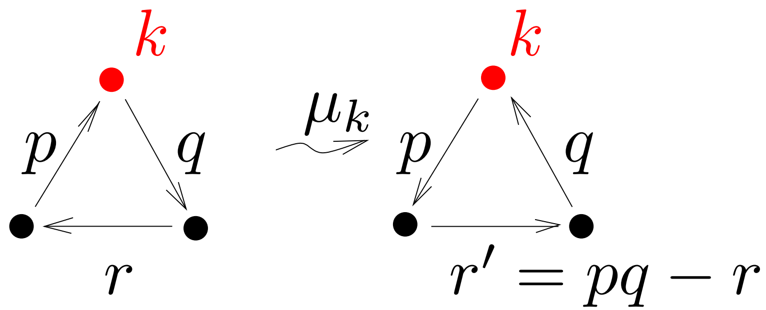
- Quiver is a directed graph without loops and 2-cycles.

- Mutation μ_k of quivers:
  - reverse all arrows incident to k;
  - for every oriented path through k do

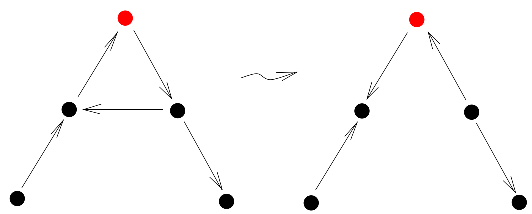


1. Quiver mutation

- Quiver is a directed graph without loops and 2-cycles.
- Mutation μ_k of quivers:
  - reverse all arrows incident to k;
  - for every oriented path through k do



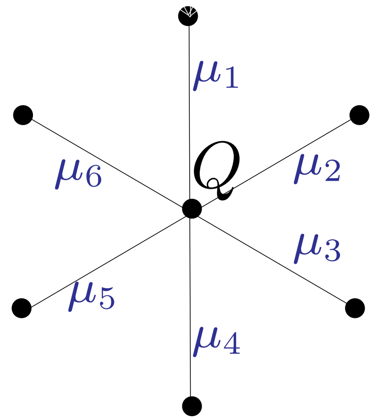
Example:



1. Quiver mutation

Iterated mutations $\longrightarrow$ many other quivers

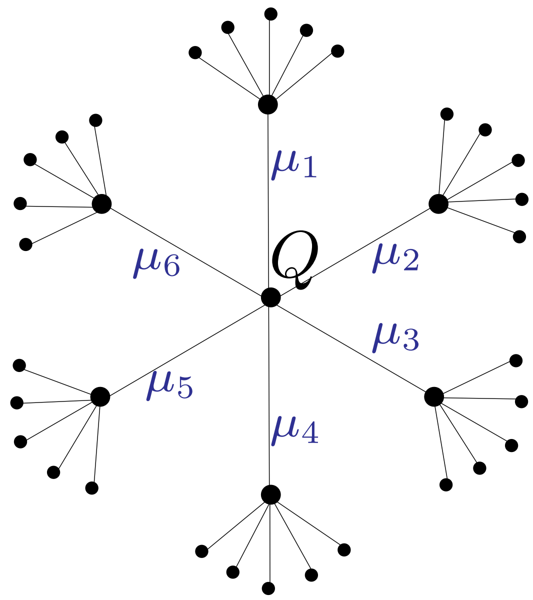
$Q \longrightarrow$ its mutation class



Property: $\mu_k \circ \mu_k(Q) = Q$ for any quiver Q.

1. Quiver mutation

$$\text{Iterated mutations} \longrightarrow \text{many other quivers}$$
$$Q \longrightarrow \text{its mutation class}$$

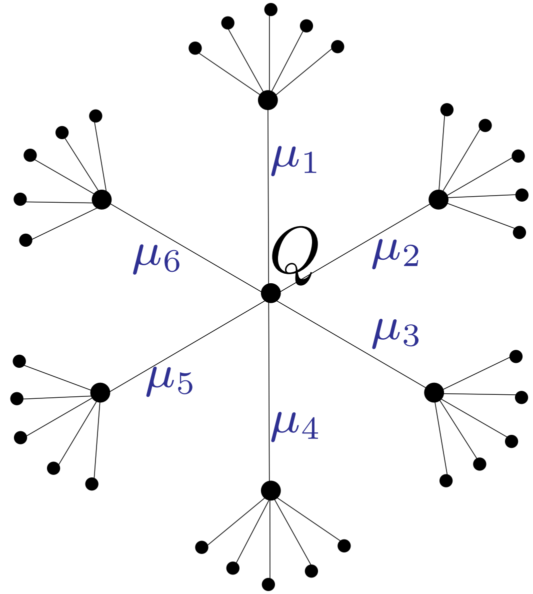


Property: $\mu_k \circ \mu_k(Q) = Q$ for any quiver Q.

1. Quiver mutation

Iterated mutations $\longrightarrow$ many other quivers

$Q \longrightarrow$ its mutation class



Property: $\mu_k \circ \mu_k(Q) = Q$ for any quiver Q.

Definition. A quiver is of finite mutation type if its mutation class contains finitely many quivers.

Question. Which quivers are of finite mutation type?

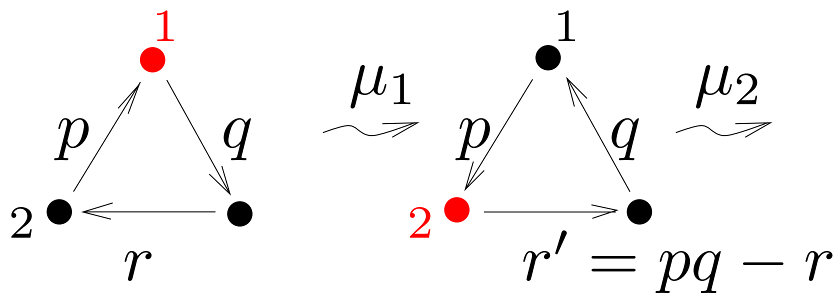
1. Quiver mutation

Question. Which quivers are of finite mutation type?

Quick answer. Not many:

If Q is connected, $|Q| \geq 3$ and Q contains arrow $\xrightarrow{p}$ with $p > 2$, then Q is mutation infinite.

Why: if $\quad q > r > 0, \quad p > 2 \qquad$ then $r' = pq - r > q > r$,
so the weghts grow under alternating mutations μ_1, μ_2.



2. Cluster algebra: seed mutation

A seed is a pair $(Q, \boldsymbol{u})$ where

Q is a quiver with $n := |Q|$ veritices,

$\boldsymbol{u} = (u_1, \ldots, u_n)$ is a set of rational functions in variables $(x_1, \ldots, x_n)$.

Initial seed: $(Q_0, \boldsymbol{u}_0)$, where $\boldsymbol{u}_0 = (x_1, \ldots, x_n)$.

2. Cluster algebra: seed mutation

A seed is a pair $(Q, \boldsymbol{u})$ where

Q is a quiver with $n := |Q|$ veritices,

$\boldsymbol{u} = (u_1, \ldots, u_n)$ is a set of rational functions in variables $(x_1, \ldots, x_n)$.

Initial seed: $(Q_0, \boldsymbol{u}_0)$, where $\boldsymbol{u}_0 = (x_1, \ldots, x_n)$.

Seed mutation: $\mu_k(Q, (u_1, \ldots, u_n)) = (\mu_k(Q), (u'_1, \ldots, u'_n))$

$$\text{where } u'_k = \frac{1}{u_k}\Big(\prod_{i \to k} u_i + \prod_{k \to j} u_j\Big)$$

$$u'_i = u_i \text{ if } i \neq k.$$

products over all incoming/outgoing arrows

2. Cluster algebra: seed mutation

A seed is a pair $(Q, \boldsymbol{u})$ where

Q is a quiver with $n := |Q|$ veritices,

$\boldsymbol{u} = (u_1, \ldots, u_n)$ is a set of rational functions

in variables $(x_1, \ldots, x_n)$.

Initial seed: $(Q_0, \boldsymbol{u}_0)$, where $\boldsymbol{u}_0 = (x_1, \ldots, x_n)$.

Seed mutation: $\mu_k(Q, (u_1, \ldots, u_n)) = (\mu_k(Q), (u'_1, \ldots, u'_n))$

$$\text{where } u'_k = \frac{1}{u_k}\Big(\prod_{i \to k} u_i + \prod_{k \to j} u_j\Big)$$

products over all
incoming/outgoing arrows

$$u'_i = u_i \text{ if } i \neq k.$$

Cluster variable: a function u_i in one of the seeds.

Cluster algebra: $\mathbb{Q}$-subalgebra of $\mathbb{Q}(x_1, \ldots, x_n)$ generated by all cluster variables.

2. Cluster algebra: finite type

A cluster algebra is of finite type

if it contains finitely many cluster variables.

Theorem. (Fomin, Zelevinsky' 2002)

A cluster algebra $\mathcal{A}(Q)$ is of finite type iff

Q is mutation-equivalent to an orientation

of a Dynkin diagram A_n, D_n, E_6, E_7, E_8.

2. **Cluster algebra:** finite type

A cluster algebra is of finite type

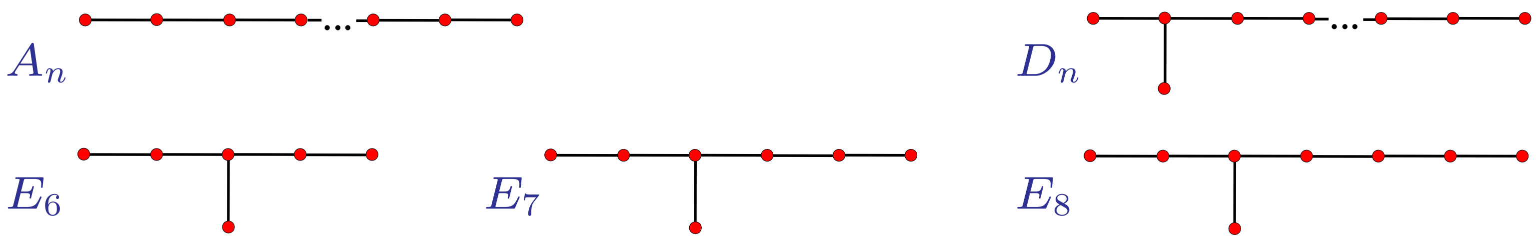
if it contains finitely many cluster variables.

Theorem. (Fomin, Zelevinsky' 2002)

A cluster algebra $\mathcal{A}(Q)$ is of finite type iff

Q is mutation-equivalent to an orientation

of a Dynkin diagram A_n, D_n, E_6, E_7, E_8.

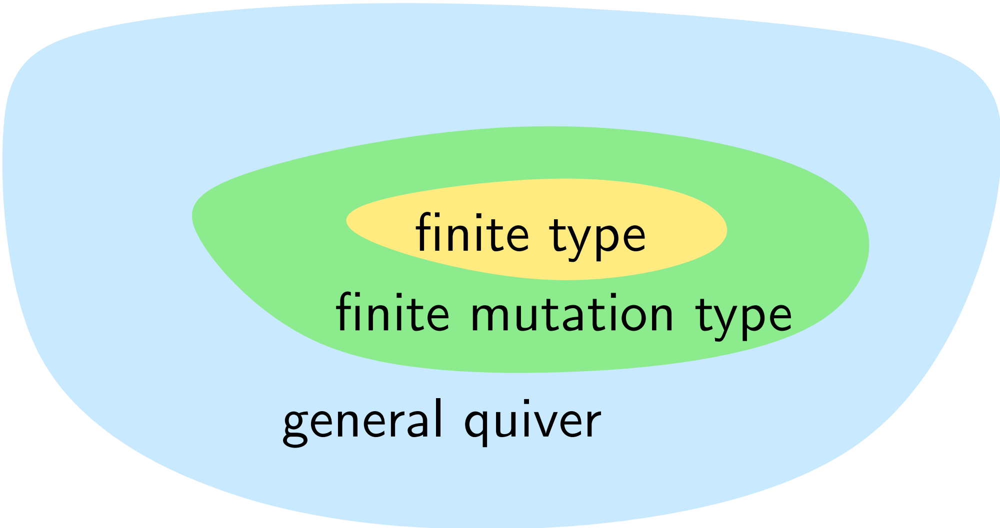


Note: Dynkin diagrams describe:

finite reflection groups, semisimple Lie algebras, surface singularities...

2. Cluster algebra: finite mutation type

A cluster algebra $\mathcal{A}(Q)$ is of finite mutation type
if Q is of finite mutation type.



mutation class	cluster variables
$< \infty$	$< \infty$
$< \infty$	∞
∞	∞
$\cancel{\infty}$	$\cancel{< \infty}$

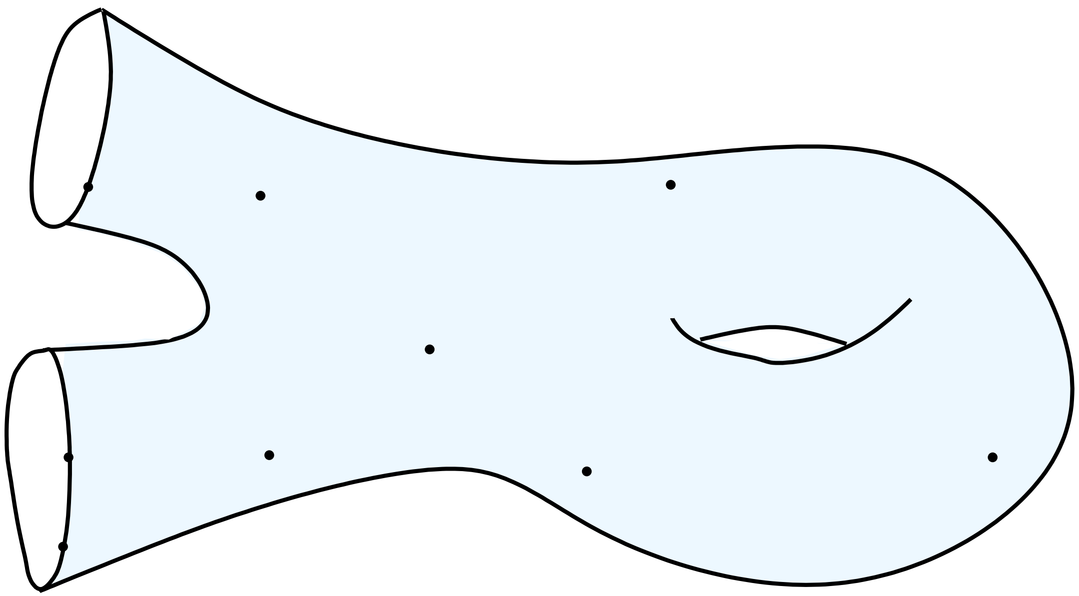
3. Finite mutation type: examples

1. $n = 2$.

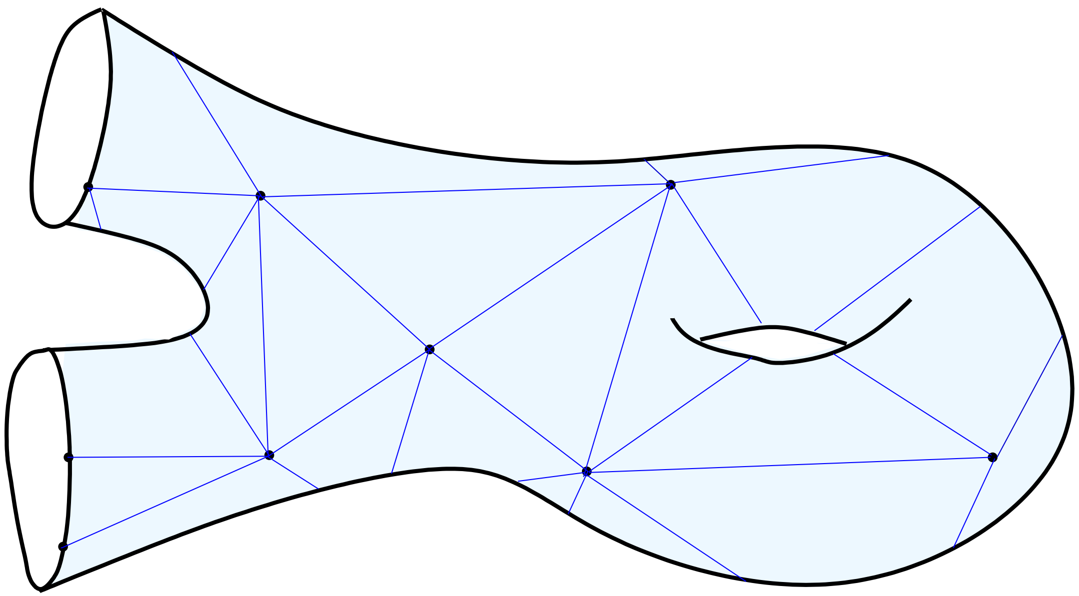
2. Quivers arising from triangulated surfaces.

3. Finitely many except that.
   (conjectured by Fomin, Shapiro, Thurston)

4. Quivers from triangulated surfaces

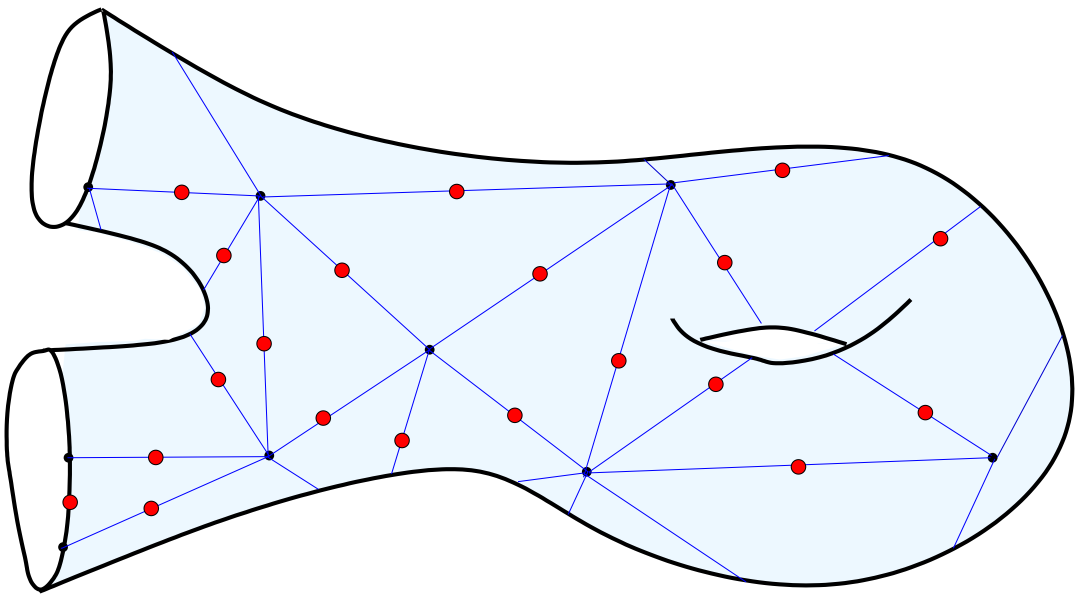


4. Quivers from triangulated surfaces



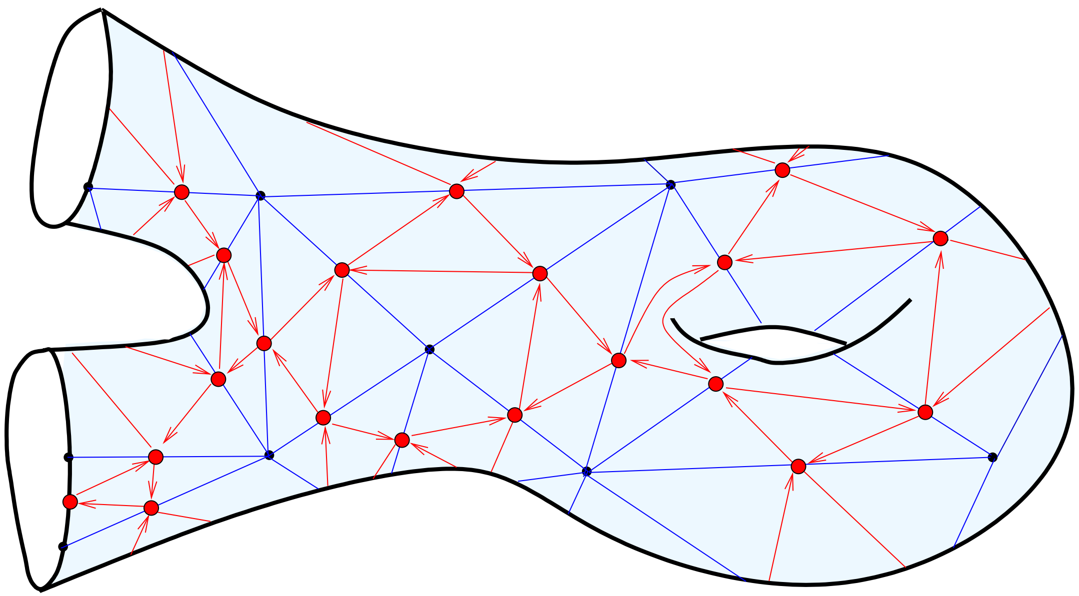
4. Quivers from triangulated surfaces

Triangulated surface $\longrightarrow$ Quiver

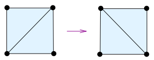


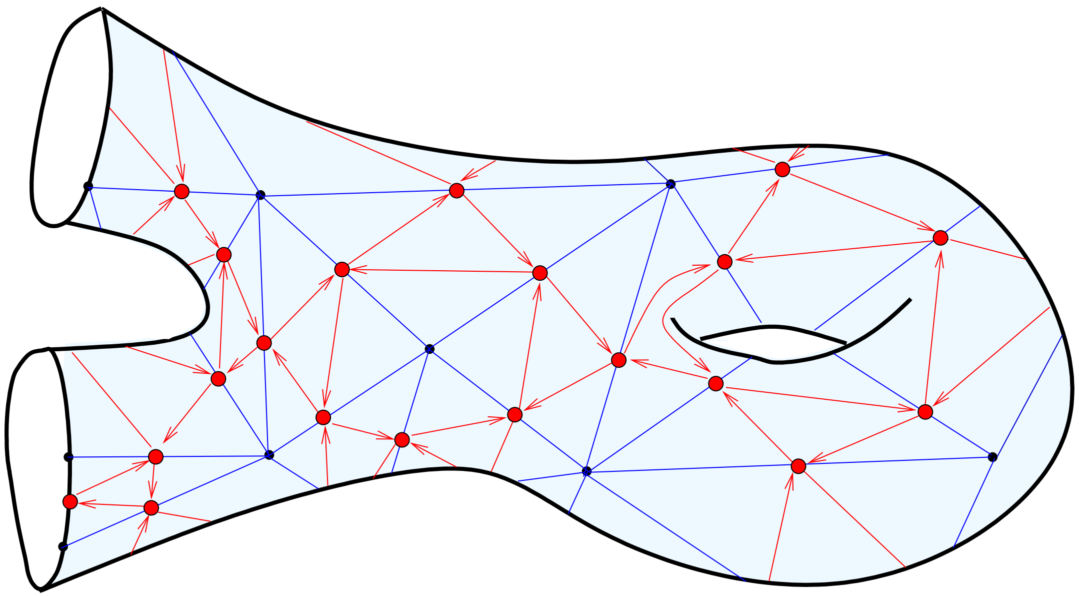
4. Quivers from triangulated surfaces

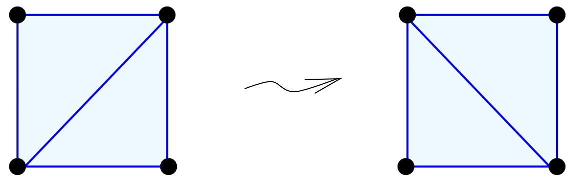
Triangulated surface	$\longrightarrow$	Quiver
edge of triangulation		vertex of quiver
two edges of one triangle		arrow of quiver



4. Quivers from triangulated surfaces

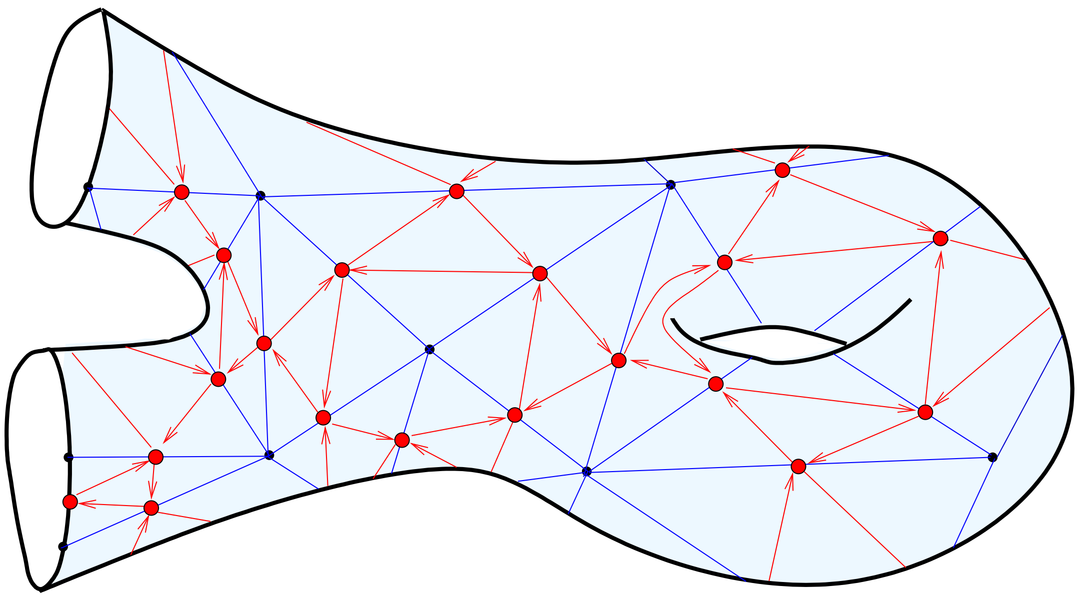
Triangulated surface	$\longrightarrow$	Quiver
edge of triangulation		vertex of quiver
two edges of one triangle		arrow of quiver
flip of triangulation 		

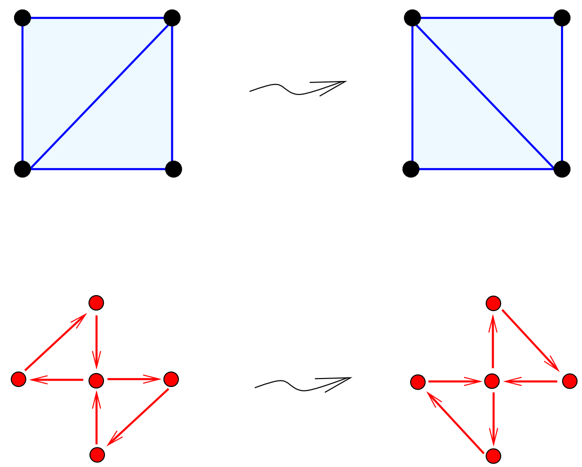




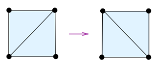
4. Quivers from triangulated surfaces

Triangulated surface	$\longrightarrow$	Quiver
edge of triangulation		vertex of quiver
two edges of one triangle		arrow of quiver
flip of triangulation		mutation of quiver





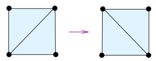
4. Quivers from triangulated surfaces

Triangulated surface	$\longrightarrow$	Quiver
edge of triangulation		vertex of quiver
two edges of one triangle		arrow of quiver
flip of triangulation 		mutation of quiver

Remark. Q from a triangulation $\Rightarrow$ weights of arrows ≤ 2.

(as every arc lies at most in two triangles)

4. Quivers from triangulated surfaces

Triangulated surface	$\longrightarrow$	Quiver
edge of triangulation		vertex of quiver
two edges of one triangle		arrow of quiver
flip of triangulation 		mutation of quiver

Remark. Q from a triangulation $\Rightarrow$ weights of arrows ≤ 2.

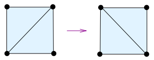
(as every arc lies at most in two triangles)

Theorem. (Hatcher, Harer) Every two triangulations of the same surface are connected by a sequence of flips.

Corollary. (a) Quivers from triangulations of the same surface are mutation-equivalent (and form the whole mutation class).

(b) Quivers from triangulations are mutation-finite.

4. Quivers from triangulated surfaces

Triangulated surface	$\longrightarrow$	Quiver
edge of triangulation		vertex of quiver
two edges of one triangle		arrow of quiver
flip of triangulation 		mutation of quiver

Remark. Q from a triangulation $\Rightarrow$ weights of arrows ≤ 2.

(as every arc lies at most in two triangles)

Theorem. (Hatcher, Harer) Every two triangulations of the same surface are connected by a sequence of flips.

Corollary. (a) Quivers from triangulations of the same surface are mutation-equivalent (and form the whole mutation class).

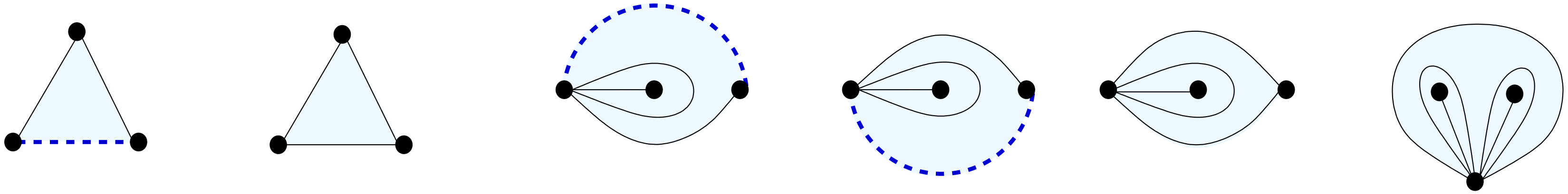
(b) Quivers from triangulations are mutation-finite.

Question. What else is mutation-finite?

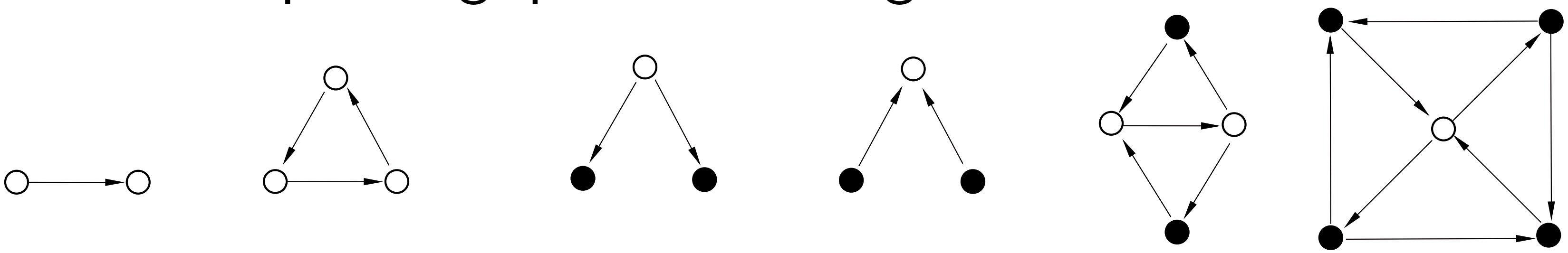
4. Quivers from triangulations: description

(Fomin-Shapiro-Thurston)

Any triangulated surface can be glued of:



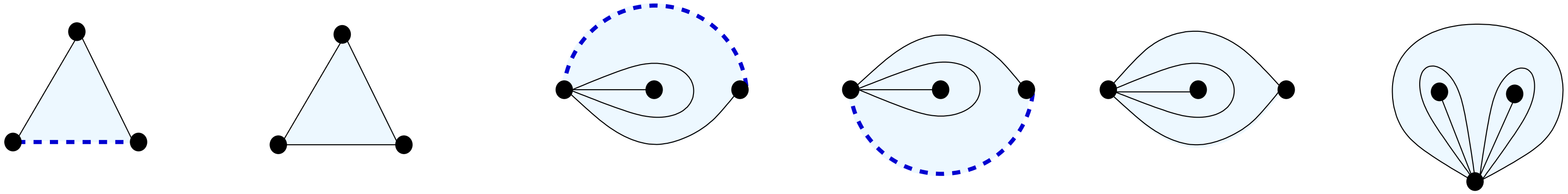
The corresponding quiver can be glued of blocks:



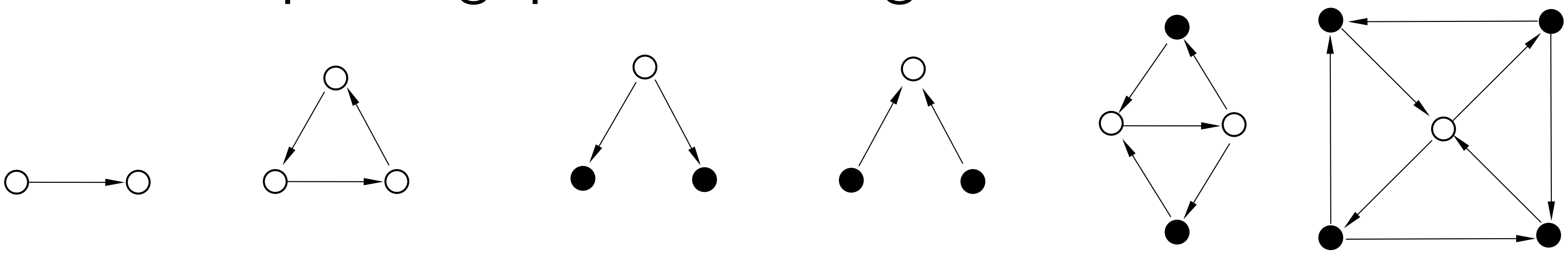
4. Quivers from triangulations: description

(Fomin-Shapiro-Thurston)

Any triangulated surface can be glued of:



The corresponding quiver can be glued of blocks:



Proposition. (Fomin-Shapiro-Thurston)

$\{Q$ is from triangulation $\} \Leftrightarrow \{Q$ is block-decomposable $\}$

Question: How to find all mutation-finite but not block-decomposable quivers?

4. Quivers from triangulations: description

(Fomin-Shapiro-Thurston)

Question: How to find all mutation-finite
but not block-decomposable quivers?

4. Quivers from triangulations: description

(Fomin-Shapiro-Thurston)

Question: How to find all mutation-finite but not block-decomposable quivers?

Interlude:

How to to classify discrete reflection groups in hyperbolic space?

1. They correspond to some polytopes (described by some diagrams);
2. Combinatorics of these polytopes is described by:
   a. subdiagrams corresponding to finite subgroups (classified) ;
   b. minimal subdiagrams correponding to infinite subgroups.

4. Quivers from triangulations: description

(Fomin-Shapiro-Thurston)

Question: How to find all mutation-finite
but not block-decomposable quivers?

Interlude:

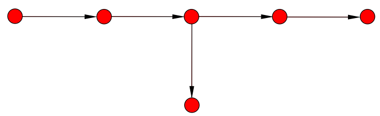
How to to classify discrete reflection groups in hyperbolic space?

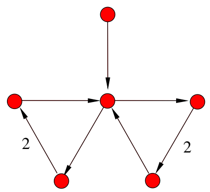
1. They correspond to some polytopes (described by some diagrams);
2. Combinatorics of these polytopes is described by:
   a. subdiagrams corresponding to finite subgroups (classified) ;
   b. minimal subdiagrams correponding to infinite subgroups.

Idea: Classify minimal non-decomposable quivers.

Lemma 1. If Q is a minimal non-decomposable quiver then $|Q| \leq 7$.

Lemma 2. If Q is a minimal non-decomposable mutation-finite quiver then is mutation equivalent to one of



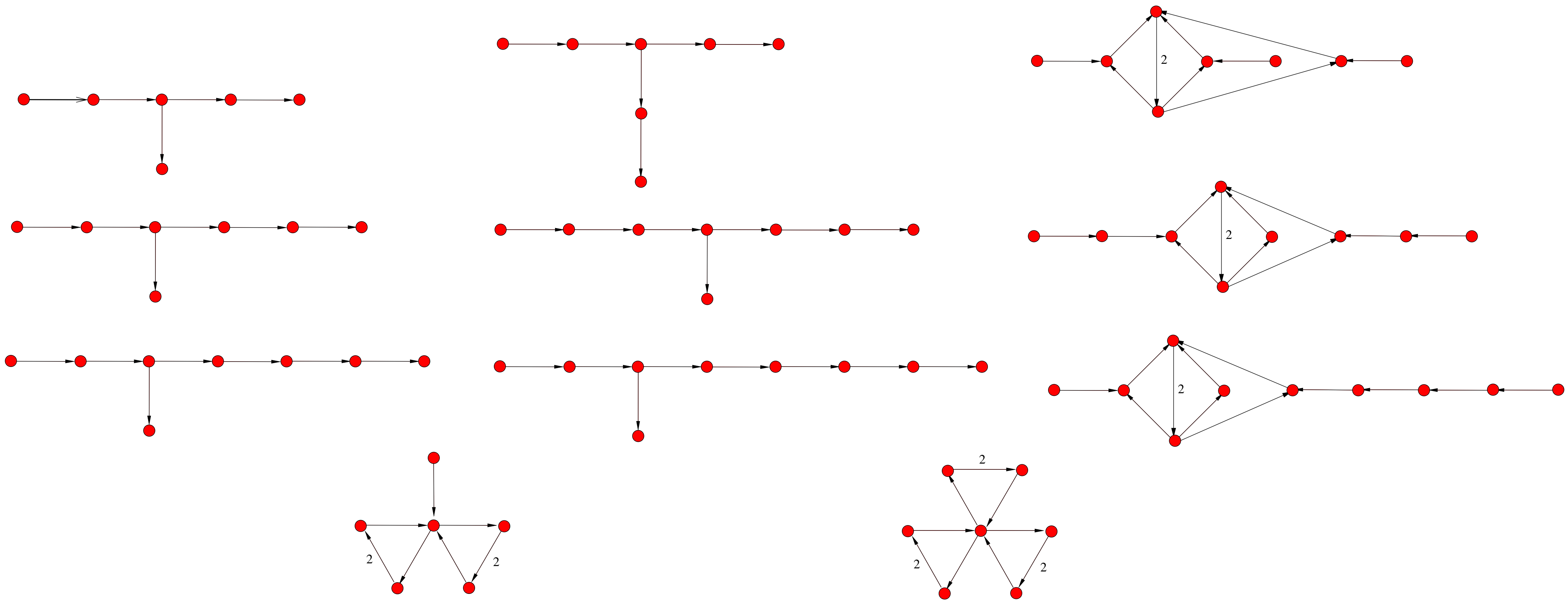


Now: - add vertices to these quivers (and their mutations) one by one
- check the obtained quiver is still mutation-finite.

Theorem 1. (A.F, M.Shapiro, P.Tumarkin' 2008)

Let Q be a connected quiver of finite mutation type. Then

- either $|Q| = 2$;
- or Q is obtained from a triangulated surface;
- or Q is mut.-equivalent to one of the following 11 quivers:



Proof:

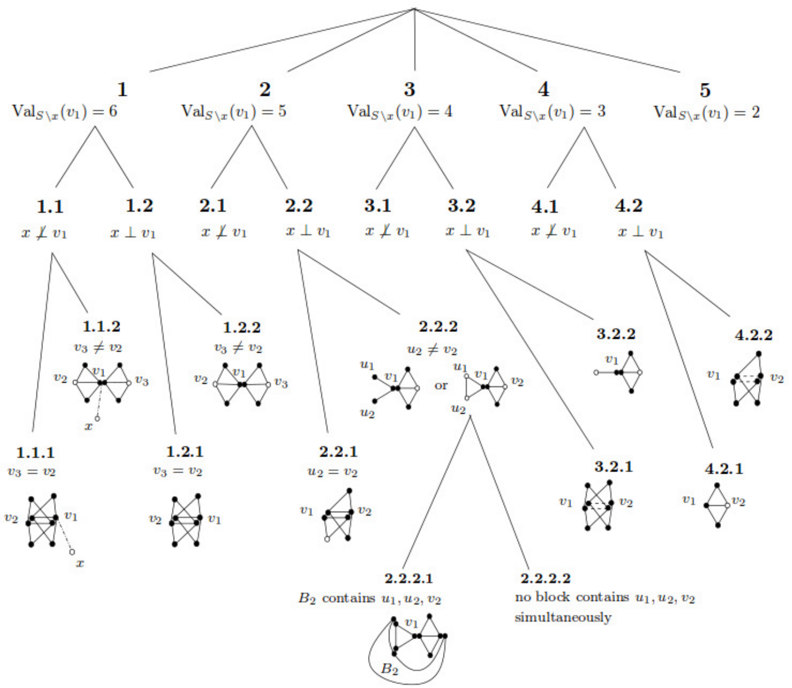
Proof: **terrible, technical ....** -but follows the same steps
as some classifications of reflection groups

Proof: **terrible, technical ....** -but follows the same steps as some classifications of reflection groups

Example. Logic scheme for a proof of some small lemma:

22 ANNA FELIKSON, MICHAEL SHAPIRO, AND PAVEL TUMARKIN

TABLE 5.1. To the proof of Lemma 5.5



Proof: **terrible, technical ....** -but follows the same steps
as some classifications of reflection groups

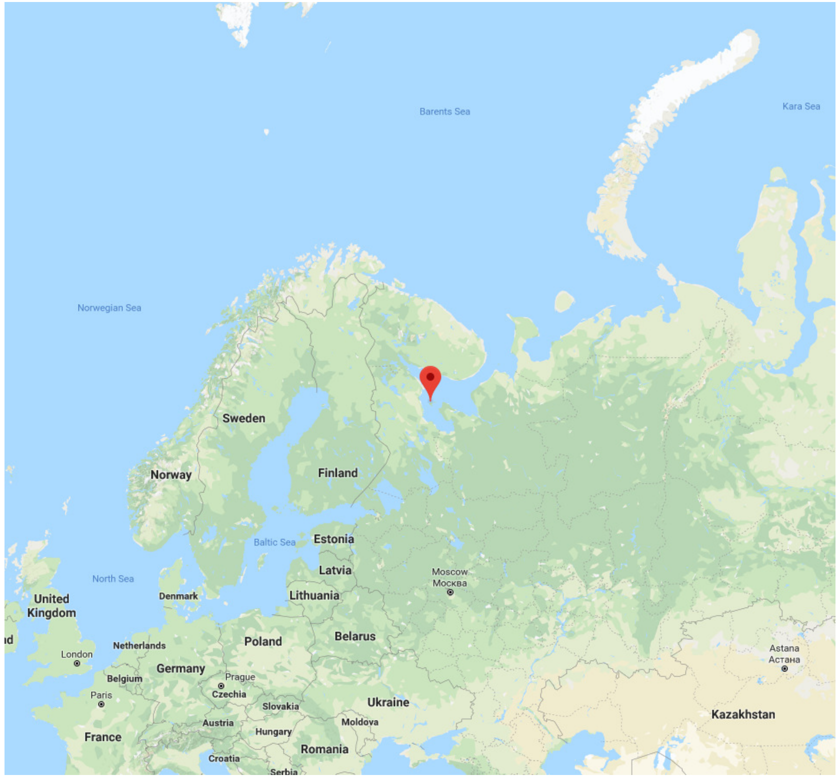
Andrei Zelevinsky: “For this sort of proofs the authors should be sent to Solovki”

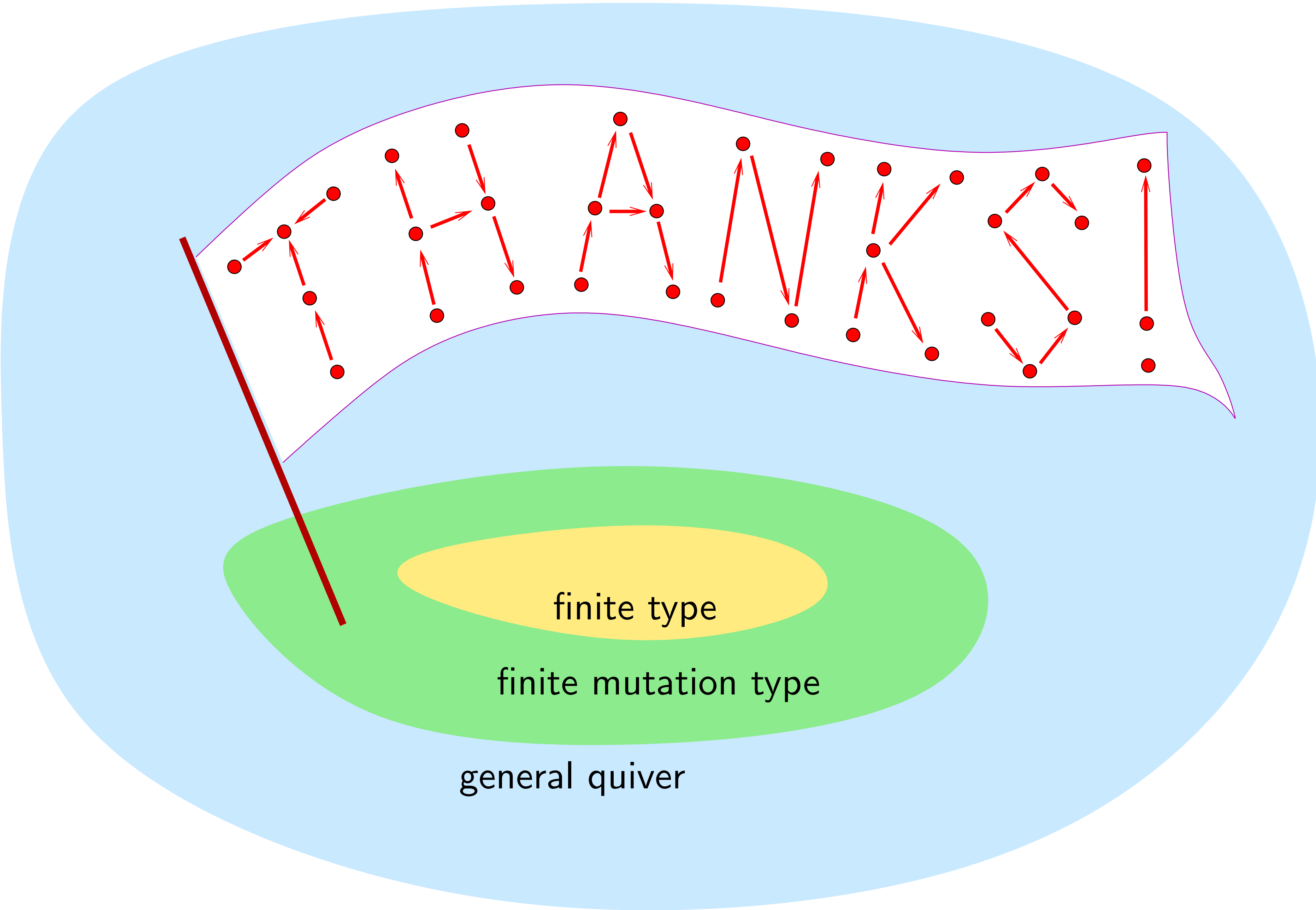
Proof: **terrible, technical ....** -but follows the same steps
as some classifications of reflection groups

Andrei Zelevinsky: "For this sort of proofs the authors should be sent to Solovki"

Proof: **terrible, technical ....** -but follows the same steps
as some classifications of reflection groups

Andrei Zelevinsky: “For this sort of proofs the authors should be sent to Solovki”




THANKS!
finite type
finite mutation type
general quiver