ALGEBRAICITY OF SPECIAL L-VALUES ATTACHED TO SIEGEL-JACOBI MODULAR FORMS

THANASIS BOUGANIS AND JOLANTA MARZEC

In this work we obtain algebraicity results on special L-values attached to Siegel-Jacobi modular forms in the spirit of Deligne's Period Conjectures. Our method relies on a generalization of the doubling method to the Jacobi group obtained in our previous work, and on introducing a notion of nearly holomorphicity for Siegel-Jacobi modular forms.

1. Introduction

This paper should be seen as a continuation of our earlier paper [3] on properties of the standard L-function attached to a Siegel-Jacobi modular form. Indeed, in [3] we have established various analytic properties (Euler product decomposition, analytic continuation and detection of poles) of the standard L-function attached to Siegel-Jacobi modular forms, and in this paper we turn our attention to algebraicity properties of some special L-values.

Shintani was the first one to attach an L-function to a Siegel-Jacobi modular form which is an eigenfunction of a properly defined Hecke algebra. He initiated the study of its analytic properties by finding an integral representation. His work was left unpublished, but then was took over by Murase [8, 9] and Arakawa [1] who obtained results on the analytic properties of this L-function using variants of the doubling method. In our previous work [3] we extended their results to a very general setting: non-trivial level, character and a totally real algebraic number field. For this purpose we applied the doubling method to the Jacobi group, and consequently related Siegel-type Jacobi Eisenstein series to the standard L-function. This identity has a further application in the current paper.

Here the starting point of our investigation is a result of Shimura in [11] on the arithmeticity of Siegel-Jacobi modular forms. Namely, if we let S be a positive definite half-integral l by l symmetric matrix, and write $M^n_{k,S}$ for the space of Siegel-Jacobi modular forms of weight k and index S (see next section for a definition), and of any congruence subgroup, and we also denote by $M^n_{k,S}(K)$ the subspace of $M^n_{k,S}$ consisting of those functions whose Fourier expansion at infinity has Fourier coefficients in a subfield K of $\mathbb{C}$, then it is shown in (loc. cit.) that $M^n_{k,S}(K) = M^n_{k,S}(\mathbb{Q}) \otimes_{\mathbb{Q}} K$.

2010 *Mathematics Subject Classification.* 11R42, 11F50, 11F66, 11F67 (primary), and 11F46 (secondary).

Key words and phrases. Jacobi group, Siegel-Jacobi modular forms, L-functions, Eisenstein series, Hecke operators.

The authors acknowledge support from EPSRC through the grant EP/N009266/1, Arithmetic of automorphic forms and special L-values.

In particular, for a given $f \in M^n_{k,S}$ and a $\sigma \in Aut(\mathbb{C}/\mathbb{Q})$ one can define the element $f^\sigma \in M^n_{k,S}$ by letting σ act on the Fourier coefficients of f.

The main result of this paper is Theorem 4.6. Without going into too much details, it may be vaguely stated as follows. Denote by $L(s, f, \chi)$ the standard L-function attached to a Siegel-Jacobi cuspidal eigenform $f \in M^n_{k,S}(\overline{\mathbb{Q}})$, which is twisted by a Dirichlet character χ, and let

$$\mathbf{\Lambda}(s, f, \chi) := L(2s - n - l/2, f, \chi) \begin{cases} L_{\mathfrak{c}}(2s - l/2, \chi\psi_S) & \text{if } l \in 2\mathbb{Z}, \\ 1 & \text{if } l \notin 2\mathbb{Z}, \end{cases}$$

where ψ_S is the non-trivial quadratic character attached to the extension $K_S := \mathbb{Q}(\sqrt{(-1)^{l/2}\det(2S)})$ if $K_S \neq \mathbb{Q}$, and otherwise $\psi_S = 1$. Then for certain integers σ and for $k > 2n + l + 1$,

$$\frac{\mathbf{\Lambda}(\sigma/2, f, \chi)}{\pi^{e_\sigma} < f, f >} \in \overline{\mathbb{Q}},$$

for an explicit power $e_\sigma \in \mathbb{N}$, and where $< f, f >$ is a Petersson inner product on the space of cuspidal Siegel-Jacobi modular forms.

To the best of our knowledge these are first results concerning algebraic properties of the special L-values of Siegel-Jacobi modular forms.

Of course, results of the above form have been proven by many researchers (most profoundly by Shimura, see for example [13]) in the cases when the standard L-function is attached to an automorphic form (e.g. a Siegel or Hermitian modular form) associated to a Shimura variety. Then such results can be also understood in the general framework of Deligne's Period Conjectures for critical values of motives [4]. Indeed, according to the general Langlands conjectures, the standard L-functions of automorphic forms related to Shimura varieties can be identified with motivic L-functions, and hence the algebraicity results for the special values of the automorphic L-functions can be also seen as a confirmation of Deligne's Period Conjecture, albeit it is usually hard to actually show that the conjectural motivic period agrees with the automorphic one.

However, Siegel-Jacobi modular forms and - in particular - the algebraicity results obtained in this paper do not fit in this framework. Indeed, since the Jacobi group is not reductive, it does not satisfy the necessary properties to be associated with a Shimura variety, and hence we are not in the situation described in the previous paragraph. Nevertheless, the Jacobi group can be actually associated with a geometric object, namely with a mixed Shimura variety, as it is explained for example in [6, 7]. Of course, we cannot expect that the standard L-function studied here can be in general identified with a motivic one. However, it is very tempting to speculate that it could be identified with an L-function of a mixed motive, and hence the theorem above could be seen as a confirmation of the generalization of Deligne's Period Conjecture to the mixed setting as for example stated by Scholl in [10].

Finally, we would like to point out that even though in some cases one can identify the standard L-function associated to a Siegel-Jacobi form with the standard L-function associated to a Siegel modular form (see for example the remark on page 252 in [9]), this is possible under some quite restrictive conditions on both index and level of the Siegel-Jacobi form. Actually, even in the situation of classical Jacobi forms this correspondence becomes quite complicated when one considers an index different than 1 and/or non-trivial level, which is very clear for example in the work of [14].

Remark: In an earlier version of [3], which one can find on the arXiv ([2]), we had also included the results of this paper. However this had resulted in a rather long exposition, and for this reason we decided to keep the two main results of our investigations separately. Namely, [3] contains now our results towards the analytic properties of the standard L function, whereas this paper focuses on the algebraic properties.

2. Preliminaries

2.1. **Siegel-Jacobi modular forms.** In this section we recall basic facts regarding Siegel-Jacobi modular forms of higher index and set up the notation. We follow closely our previous work [3].

Let F be a totally real algebraic number field of degree d, $\mathfrak{d}$ the different of F, and $\mathfrak{o}$ its ring of integers. For two natural numbers l, n, we consider the Jacobi group $\boldsymbol{G} := \boldsymbol{G}^{n,l} := H^{n,l} \rtimes \mathrm{Sp}_n$ of degree n and index l over F:

$$\boldsymbol{G}^{n,l}(F) := \{\boldsymbol{g} = (\lambda, \mu, \kappa)g : \lambda, \mu \in M_{l,n}(F), \kappa \in Sym_l(F), g \in G^n(F)\},$$

where $H(F) := H^{n,l}(F) := \{(\lambda, \mu, \kappa)1_{2n} \in \boldsymbol{G}^{n,l}(F)\}$ is the Heisenberg group, and

$$G^n(F) := \mathrm{Sp}_n(F) := \left\{g \in \mathrm{SL}_{2n}(F) \colon {}^tg \left(\begin{smallmatrix} & -1_n \\ 1_n & \end{smallmatrix}\right) g = \left(\begin{smallmatrix} & -1_n \\ 1_n & \end{smallmatrix}\right)\right\}.$$

The group law is given by

$$(\lambda, \mu, \kappa)g(\lambda', \mu', \kappa')g' := (\lambda + \tilde{\lambda}, \mu + \tilde{\mu}, \kappa + \kappa' + \lambda\,{}^t\tilde{\mu} + \tilde{\mu}\,{}^t\lambda + \tilde{\lambda}\,{}^t\tilde{\mu} - \lambda'\,{}^t\mu')gg',$$

where $(\tilde{\lambda}\;\tilde{\mu}) := (\lambda'\;\mu')g^{-1} = (\lambda'\,{}^td - \mu'\,{}^tc \quad \mu'\,{}^ta - \lambda'\,{}^tb)$, and the identity element of $\boldsymbol{G}^{n,l}(F)$ is $1_H 1_{2n}$, where $1_H := (0,0,0)$ is the identity element of $H^{n,l}(F)$ (whenever it does not lead to any confusion we suppress the indices n, l). For an element $g \in \mathrm{Sp}_n$ we write $g = \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix}$, where $a_g, b_g, c_g, d_g \in M_n$.

We write $\{\sigma_v : F \hookrightarrow \mathbb{R},\ v \in \mathbf{a}\}$ for the set of real embeddings of F, $\mathbf{a}$ denoting the set of archimedean places of F. Each σ_v induces an embedding $\boldsymbol{G}(F) \hookrightarrow \boldsymbol{G}(\mathbb{R})$; we will write $(\lambda_v, \mu_v, \kappa_v)g_v$ for $\sigma_v(\boldsymbol{g})$. The group $\boldsymbol{G}(\mathbb{R})^{\mathbf{a}}$ acts on $\mathcal{H}_{n,l} := (\mathbb{H}_n \times M_{l,n}(\mathbb{C}))^{\mathbf{a}}$ component wise via

$$\boldsymbol{g}z = \boldsymbol{g}(\tau, w) = (\lambda, \mu, \kappa)g(\tau, w) = \prod_{v \in \mathbf{a}} (g_v\tau_v, w_v\lambda(g_v, \tau_v)^{-1} + \lambda_v g_v \tau_v + \mu_v),$$

where $g_v\tau_v = (a_v\tau_v + b_v)(c_v\tau_v + d_v)^{-1}$ and $\lambda(g_v, \tau_v) := (c_v\tau_v + d_v)$ for $g_v = \left(\begin{smallmatrix} a_v & b_v \\ c_v & d_v \end{smallmatrix}\right)$.

For $k \in \mathbb{Z}^{\mathbf{a}}$ and a matrix $S \in Sym_l(\mathfrak{d}^{-1})$ we define the factor of automorphy of weight k and index S by

$$J_{k,S} \colon \boldsymbol{G}^{n,l}(F) \times \mathcal{H}_{n,l} \to \mathbb{C}$$

$$J_{k,S}(\boldsymbol{g}, z) = J_{k,S}(\boldsymbol{g}, (\tau, w)) := \prod_{v \in \mathbf{a}} j(g_v, \tau_v)^{k_v} \mathcal{J}_{S_v}(\boldsymbol{g_v}, \tau_v, w_v),$$

where $\boldsymbol{g} = (\lambda, \mu, \kappa)g$, $j(g_v, \tau_v) = \det(c_v\tau_v + d_v) = \det(\lambda(g_v, \tau_v))$ and

$$\mathcal{J}_{S_v}(\boldsymbol{g_v}, \tau_v, w_v) = e(-\mathrm{tr}\,(S_v\kappa_v) + \mathrm{tr}\,(S_v[w_v]\lambda(g_v,\tau_v)^{-1}c_v) \\ - 2\mathrm{tr}\,({}^t\lambda_v S_v w_v \lambda(g_v,\tau_v)^{-1}) - \mathrm{tr}\,(S_v[\lambda_v]g_v\tau_v))$$

with $e(x) := e^{2\pi i x}$, and we set $S[x] := {}^t x S x$; $J_{k,S}$ satisfies the usual cocycle relation:

$$J_{k,S}(\boldsymbol{g}\boldsymbol{g}', z) = J_{k,S}(\boldsymbol{g}, \boldsymbol{g}'\, z) J_{k,S}(\boldsymbol{g}', z). \tag{1}$$

For a function $f\colon \mathcal{H}_{n,l} \to \mathbb{C}$ we define

$$(f|_{k,S}\, \boldsymbol{g})(z) := J_{k,S}(\boldsymbol{g}, z)^{-1} f(\boldsymbol{g}\, z). \tag{2}$$

A subgroup $\boldsymbol{\Gamma}$ of $\boldsymbol{G}(F)$ will be called a congruence subgroup if there exist a fractional ideal $\mathfrak{b}$ and an integral ideal $\mathfrak{c}$ of F such that $\boldsymbol{\Gamma}$ is a subgroup of finite index of the group $G(F) \cap \boldsymbol{g}K[\mathfrak{b},\mathfrak{c}]\boldsymbol{g}^{-1}$ for some $\boldsymbol{g} \in \boldsymbol{G}_{\mathbf{h}} := \prod_{v\in\mathbf{h}} \boldsymbol{G}(F_v)$, $\mathbf{h}$ denoting non-archimedean places of F. The group $K[\mathfrak{b},\mathfrak{c}]$ is defined as $K[\mathfrak{b},\mathfrak{c}] := K_{\mathbf{h}}[\mathfrak{b},\mathfrak{c}]\boldsymbol{G}_{\mathbf{a}}$, where $\boldsymbol{G}_{\mathbf{a}} = \prod_{v\in\mathbf{a}} \boldsymbol{G}(F_v)$, and

$$K_{\mathbf{h}}[\mathfrak{b},\mathfrak{c}] := C_{\mathbf{h}}[\mathfrak{o},\mathfrak{b}^{-1},\mathfrak{b}^{-1}] \rtimes D_{\mathbf{h}}[\mathfrak{b}^{-1},\mathfrak{b}\mathfrak{c}] \subset \boldsymbol{G}_{\mathbf{h}},$$

$$C_{\mathbf{h}}[\mathfrak{o},\mathfrak{b}^{-1},\mathfrak{b}^{-1}] := \{(\lambda,\mu,\kappa) \in {\prod_{v\in\mathbf{h}}}' H(F_v) : \forall v \in \mathbf{h} \quad \substack{\lambda_v \in M_{l,n}(\mathfrak{o}_v),\ \mu_v \in M_{l,n}(\mathfrak{b}_v^{-1}), \\ \kappa_v \in Sym_l(\mathfrak{b}_v^{-1})}\},$$

$$D_{\mathbf{h}}[\mathfrak{b}^{-1},\mathfrak{b}\mathfrak{c}] := \prod_{v\in\mathbf{h}} D_v[\mathfrak{b}^{-1},\mathfrak{b}\mathfrak{c}],$$

$$D_v[\mathfrak{b}^{-1},\mathfrak{b}\mathfrak{c}] := \left\{ x = \begin{pmatrix} a_x & b_x \\ c_x & d_x \end{pmatrix} \in G_v : \substack{a_x \in M_n(\mathfrak{o}_v),\quad b_x \in M_n(\mathfrak{b}_v^{-1}), \\ c_x \in M_n(\mathfrak{b}_v\mathfrak{c}_v),\quad d_x \in M_n(\mathfrak{o}_v)} \right\}.$$

We now consider an $S \in \mathfrak{b}\mathfrak{d}^{-1}\mathcal{T}_l$ where

$$\mathcal{T}_l := \{x \in Sym_l(F) : \mathrm{tr}\,(xy) \in \mathfrak{o} \text{ for all } y \in Sym_l(\mathfrak{o})\}, \tag{3}$$

and assume additionally that S is positive definite in the sense that if we write $S_v := \sigma_v(S) \in Sym_l(\mathbb{R})$ for $v \in \mathbf{a}$, then all S_v are positive definite.

Definition 2.1. Let k and S be as above, and $\boldsymbol{\Gamma}$ a congruence subgroup equipped with a homorphism χ. A Siegel-Jacobi modular form of weight $k \in \mathbb{Z}^{\mathbf{a}}$, index S, level $\boldsymbol{\Gamma}$ and Nebentypus χ is a holomorphic function $f\colon \mathcal{H}_{n,l} \to \mathbb{C}$ such that

(1) $f|_{k,S}\, \boldsymbol{g} = \chi(\boldsymbol{g})f$ for every $\boldsymbol{g} \in \boldsymbol{\Gamma}$,
(2) for each $g \in G^n(F)$, $f|_{k,S}\, g$ admits a Fourier expansion of the form

$$f|_{k,S}\, g(\tau,w) = \sum_{\substack{t\in L \\ t\geq 0}} \sum_{r\in M} c(\boldsymbol{g};t,r)\mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,(t\tau))\mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,({}^t r w)) \quad (*)$$

for some appropriate lattices $L \subset Sym_n(F)$ and $M \subset M_{l,n}(F)$, where $t \geq 0$ means that t_v is semi-positive definite for each $v \in \mathbf{a}$.

We will denote the space of such functions by $M^n_{k,S}(\boldsymbol{\Gamma},\chi)$.

We say that f is a cusp form if in the expansion $(*)$ above for every $g \in G^n(F)$, we have $c(\mathbf{g};t,r) = 0$ unless $\begin{pmatrix} S_v & r_v \\ {}^t r_v & t_v \end{pmatrix}$ is positive definite for every $v \in \mathbf{a}$. The space of cusp forms will be denoted by $S^n_{k,S}(\boldsymbol{\Gamma},\chi)$.

We define Petersson inner product of Siegel-Jacobi forms f and g of weight k and level $\mathbf{\Gamma}$ under assumption that one of them is a cusp form as:

$$< f,g >:= vol(A)^{-1}\int_A f(z)\overline{g(z)}\Delta_{S,k}(z)dz, \quad A := \mathbf{\Gamma}\setminus\mathcal{H}_{n,l},$$

where for $z = (\tau, w) \in \mathcal{H}_{n,l}$, $\tau = x+iy$ with $x, y \in Sym_n(F_{\mathbf{a}})$ and $w = u+iv$ with $u, v \in M_{l,n}(F_{\mathbf{a}})$, we set $dz := d(\tau,w) := \det(y)^{-(l+n+1)}dxdydudv$ and $\Delta_{S,k}(z) := \det(y)^k\mathbf{e}_{\mathbf{a}}(-4\pi\mathrm{tr}\,({}^tvSvy^{-1}))$. In this way the inner product is independent of the group $\mathbf{\Gamma}$.

2.2. **Adelic Siegel-Jacobi modular forms.** Denote by $\mathbb{A}$ the adeles of F, and let $K_{\mathbf{h}}[\mathfrak{b},\mathfrak{c}] \subset \boldsymbol{G}_{\mathbf{h}}$ be the subgroup defined in the previous section. Then Jacobi group $\boldsymbol{G}$ satisfies strong approximation theorem:

$$\boldsymbol{G}(\mathbb{A}) = \boldsymbol{G}(F)K_{\mathbf{h}}[\mathfrak{b},\mathfrak{c}]\boldsymbol{G}_{\mathbf{a}}.$$

It will be useful to define also groups

$$K_0[\mathfrak{b},\mathfrak{c}] := K_0^n[\mathfrak{b},\mathfrak{c}] := K_{\mathbf{h}}[\mathfrak{b},\mathfrak{c}]\times K_\infty \quad \text{and} \quad K := K^n := K_{\mathbf{h}}[\mathfrak{b},\mathfrak{c}](H_{\mathbf{a}}^{n,l}\rtimes D_\infty^{\mathbf{a}}),$$

where $K_\infty \simeq Sym_l(\mathbb{R})^{\mathbf{a}}\rtimes D_\infty^{\mathbf{a}} \subset H^{n,l}(\mathbb{R})^{\mathbf{a}}\rtimes \mathrm{Sp}_n(\mathbb{R})^{\mathbf{a}}$ is the stabilizer of the point $\mathbf{i}_0 := (\boldsymbol{i}, 0) \in \mathcal{H}_{n,l}$, and D_∞ is the maximal compact subgroup of $\mathrm{Sp}_n(\mathbb{R})$. Here $\boldsymbol{i} \in \mathbb{H}_n^{\mathbf{a}}$ denotes the point $(i1_n \ldots, i1_n)$ on the Siegel upper space.

Now, we fix once and for all an additive character $\Psi : \mathbb{A}/F \to \mathbb{C}^\times$ as follows. Write $\Psi = \prod_{v\in\mathbf{h}}\Psi_v\prod_{v\in\mathbf{a}}\Psi_v$ and define

$$\Psi_v(x_v) := \begin{cases} e(-y_v), & v \in \mathbf{h} \\ e(x_v), & v\in \mathbf{a}, \end{cases}$$

where $y_v \in \mathbb{Q}$ is such that $\mathrm{Tr}_{F_v/\mathbb{Q}_p}(x_v) - y_v \in \mathbb{Z}_p$ for $p := v\cap\mathbb{Q}$. Given a symmetric matrix $S \in Sym_l(F)$ we define a character $\psi_S : Sym_l(\mathbb{A})/Sym_l(F) \to \mathbb{C}^\times$ by taking $\psi_S(\kappa) := \Psi(\mathrm{tr}\,(S\kappa)$.

Consider an adelic Hecke character $\chi : \mathbb{A}^\times/F^\times \to \mathbb{C}^\times$ of F of finite order such that $\chi_v(x) = 1$ for all $x \in \mathfrak{o}_v^\times$ with $x-1\in\mathfrak{c}_v$. We extend this character to a character of the group $K_0[\mathfrak{b},\mathfrak{c}]$ by setting $\chi(w) := \prod_{v|\mathfrak{c}}\chi_v(\det(a_g))^{-1}$ for $w = hg \in K_0[\mathfrak{b},\mathfrak{c}]$.

Now, let $k\in\mathbb{Z}^{\mathbf{a}}$ and $S \in Sym_l(F)$ be such that $S \in \mathfrak{b}\mathfrak{d}^{-1}\mathcal{T}_l$ with $\mathcal{T}_l$ as in (3). Moreover, let K be an open subgroup of $K[\mathfrak{b},\mathfrak{c}]$ for some $\mathfrak{b}$ and $\mathfrak{c}$.

Definition 2.2. An adelic Siegel-Jacobi modular form of degree n, weight k, index S and character χ, with respect to the congruence subgroup K is a function $\mathbf{f} : \boldsymbol{G}(\mathbb{A}) \to \mathbb{C}$ such that

(1) $\mathbf{f}\,((0,0,\kappa)\gamma\boldsymbol{g}w) = \chi(w)J_{k,S}(w,\mathbf{i}_0)^{-1}\psi_S(\kappa)\mathbf{f}(\boldsymbol{g})$, for all $\kappa \in Sym_l(\mathbb{A})$, $\gamma \in \boldsymbol{G}(F)$, $\boldsymbol{g} \in \boldsymbol{G}(\mathbb{A})$ and $w \in K\cap K_0[\mathfrak{b},\mathfrak{c}]$;
(2) for every $\boldsymbol{g} \in \boldsymbol{G}_{\mathbf{h}}$ the function $f_{\boldsymbol{g}}$ on $\mathcal{H}_{n,l}$ defined by the relation

$$(f_{\boldsymbol{g}}|_{k,S}\boldsymbol{y})(\mathbf{i}_0) := \mathbf{f}(\boldsymbol{g}\boldsymbol{y}) \quad \text{for all } \boldsymbol{y}\in\boldsymbol{G}_{\mathbf{a}}$$

is a Siegel-Jacobi modular form for the congruence group $\mathbf{\Gamma}^{\boldsymbol{g}} := \boldsymbol{G}(F)\cap\boldsymbol{g}K\boldsymbol{g}^{-1}$.

We denote the space of adelic Siegel-Jacobi modular forms by $\mathcal{M}^n_{k,S}(K,\chi)$. For any given $\boldsymbol{g} \in \boldsymbol{G}_{\mathbf{h}}$ there exists a bijection

$$\mathcal{M}^n_{k,S}(K,\chi) \to M^n_{k,S}(\boldsymbol{\Gamma}^{\boldsymbol{g}}, \chi_{\boldsymbol{g}}), \qquad \mathbf{f} \mapsto f_{\boldsymbol{g}}, \tag{4}$$

where $\chi_{\boldsymbol{g}}$ is the character on $\boldsymbol{\Gamma}^{\boldsymbol{g}}$ defined as $\chi_{\boldsymbol{g}}(\gamma) := \chi(\boldsymbol{g}^{-1}\gamma\boldsymbol{g})$. We say that $\mathbf{f}$ is a cusp form, and we denote this space by $\mathcal{S}^n_{k,l}(K,\chi)$, if in the above notation $f_{\boldsymbol{g}}$ is a cusp form for all $\boldsymbol{g} \in \boldsymbol{G}_{\mathbf{h}}$. If $\boldsymbol{g} = 1$, we will write f for the Siegel-Jacobi modular form corresponding to $\mathbf{f}$ via (4).

3. The standard L-function and the doubling method identity

In this section we recall some results and notation from [3] which will be necessary to establish results in the next section.

3.1. **The L-function.** We start by fixing some notation. For a fractional ideal $\mathfrak{b}$, and an integral ideal $\mathfrak{c}$ we let

$$\boldsymbol{D} := \{(\lambda,\mu,\kappa)x \in C_v[\mathfrak{o},\mathfrak{b}^{-1},\mathfrak{b}^{-1}]D_v[\mathfrak{b}^{-1}\mathfrak{c},\mathfrak{b}\mathfrak{c}] : (a_x - 1_n)_v \in M_n(\mathfrak{c}_v) \text{ for every } v|\mathfrak{c}\},$$

$$\boldsymbol{\Gamma} := \boldsymbol{\Gamma}_1(\mathfrak{c}) := \boldsymbol{G}^n(F) \cap \boldsymbol{D},$$

$$Q(\mathfrak{c}) := \{r \in \mathrm{GL}_n(\mathbb{A}_{\mathbf{h}}) \cap \prod_{v\in\mathbf{h}} M_n(\mathfrak{o}_v) : r_v = 1_n \text{ for every } v|\mathfrak{c}\},$$

$$R(\mathfrak{c}) := \{\mathrm{diag}[\tilde{r}, r] : r \in Q(\mathfrak{c})\}.$$

For $r \in Q(\mathfrak{c})$ and $f \in M^n_{k,S}(\boldsymbol{\Gamma})$ we define a linear operator $T_r : M^n_{k,S}(\boldsymbol{\Gamma}) \to M^n_{k,S}(\boldsymbol{\Gamma})$ by

$$f|T_r := \sum_{\boldsymbol{\alpha}\in\boldsymbol{A}} f|_{k,S}\boldsymbol{\alpha}, \tag{5}$$

where $\boldsymbol{A} \subset \boldsymbol{G}^n(F)$ is such that $\boldsymbol{G}^n(F) \cap \boldsymbol{D}\mathrm{diag}[\tilde{r}, r]\boldsymbol{D} = \coprod_{\boldsymbol{\alpha}\in\boldsymbol{A}} \boldsymbol{\Gamma}\boldsymbol{\alpha}$. Further, for an integral ideal $\mathfrak{a}$ of F we put

$$f|T(\mathfrak{a}) := \sum_{\substack{r\in Q(\mathfrak{c})\\ \det(r)\mathfrak{o}=\mathfrak{a}}} f|T_r,$$

where we sum over all those r for which the cosets $\prod_{v\in\mathbf{h}} \mathrm{GL}_n(\mathfrak{o}_v)\, r \prod_{v\in\mathbf{h}} \mathrm{GL}_n(\mathfrak{o}_v)$ are distinct.

Note that if $\mathbf{f}|T_r$ is the adelic Siegel-Jacobi form associated to $f|T_r$ by the bijection given in (4) with $\mathbf{g} = 1$, then

$$(\mathbf{f}|T_r)(x) = \sum_{\boldsymbol{\alpha}\in\boldsymbol{A}} \mathbf{f}(x\boldsymbol{\alpha}^{-1}), \qquad x \in \boldsymbol{G}^n(\mathbb{A}),$$

where $\boldsymbol{D}\mathrm{diag}[\tilde{r}, r]\boldsymbol{D} = \coprod_{\boldsymbol{\alpha}\in\boldsymbol{A}} \boldsymbol{D}\boldsymbol{\alpha}$ with $\boldsymbol{A} \subset \boldsymbol{G}_{\mathbf{h}}$; we define $\mathbf{f}|T(\mathfrak{a})$ in a similar way.

We now consider a nonzero $\mathbf{f} \in \mathcal{S}^n_{k,S}(\boldsymbol{D})$ such that $\mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}$ for all integral ideals $\mathfrak{a}$ of F. For a Hecke character χ of F, and denoting by χ^* the corresponding ideal character, we define an absolutely convergent series

$$D(s,\mathbf{f},\chi) := \sum_{\mathfrak{a}} \lambda(\mathfrak{a})\chi^*(\mathfrak{a})N(\mathfrak{a})^{-s}, \qquad \mathrm{Re}(s) > 2n + l + 1.$$

In [3] we proved the following theorem regarding the Euler product representation of this Dirichlet series. For the condition $M_{\mathfrak{p}}^{+}$ for primes away from $\mathfrak{c}$ we refer to [3].

Theorem 3.1 (Theorem 7.1, [3]). *Let $0 \neq \mathbf{f} \in \mathcal{S}_{k,S}^{n}(\boldsymbol{D})$ be such that $\mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}$ for all integral ideals $\mathfrak{a}$ of F. Assume that the matrix S satisfies the condition $M_{\mathfrak{p}}^{+}$ for every prime ideal $\mathfrak{p}$ with $(\mathfrak{p},\mathfrak{c}) = 1$. Then*

$$\mathfrak{L}(\chi,s)D(s+n+l/2,\mathbf{f},\chi) = L(s,\mathbf{f},\chi) := \prod_{\mathfrak{p}} L_{\mathfrak{p}}(\chi^{*}(\mathfrak{p})N(\mathfrak{p})^{-s})^{-s},$$

where for every prime ideal $\mathfrak{p}$ of F

$$L_{\mathfrak{p}}(X) = \begin{cases} \prod_{i=1}^{n}\Big((1-\mu_{\mathfrak{p},i}X)(1-\mu_{\mathfrak{p},i}^{-1}X)\Big), & \mu_{\mathfrak{p},i} \in \mathbb{C}^{\times} \quad \text{if } (\mathfrak{p},\mathfrak{c}) = 1, \\ 1 & \text{if } (\mathfrak{p},\mathfrak{c}) \neq 1. \end{cases}$$

Moreover, $\mathfrak{L}(\chi,s) = \prod_{(\mathfrak{p},\mathfrak{c})=1} \mathfrak{L}_{\mathfrak{p}}(\chi,s)$, where

$$\mathfrak{L}_{\mathfrak{p}}(\chi,s) := G_{\mathfrak{p}}(\chi,s)\cdot\begin{cases} \prod_{i=1}^{n} L_{\mathfrak{p}}(2s+2n-2i,\chi^{2}) & \text{if } l \in 2\mathbb{Z} \\ \prod_{i=1}^{n} L_{\mathfrak{p}}(2s+2n-2i+1,\chi^{2}) & \text{if } l \notin 2\mathbb{Z} \end{cases}$$

and $G_{\mathfrak{p}}(\chi,s)$ is a ratio of Euler factors which for almost all $\mathfrak{p}$ is equal to one. In particular, the function $L(s,\mathbf{f},\chi)$ is absolutely convergent for $\mathrm{Re}(s) > n + l/2 + 1$.

We note here that the Euler product expression implies that

$$L(s,\mathbf{f},\chi) \neq 0, \qquad \mathrm{Re}(s) > n + l/2 + 1. \tag{6}$$

We set $f^{c}(z) := \overline{f(-\overline{z})}$, where f corresponds to $\mathbf{f} \in S_{k,S}^{n}(\boldsymbol{D})$ via (4). We write $\mathbf{f}^{c}$ for the adelic form corresponding to f^{c}. Then

Proposition 3.2 (Proposition 7.9, [3]). *Let $f \in S_{k,S}^{n}(\boldsymbol{\Gamma})$ be an eigenform with $f|T(\mathfrak{a}) = \lambda(\mathfrak{a})f$ for all fractional ideals $\mathfrak{a}$ prime to $\mathfrak{c}$. Then so is f^{c}. In particular, $f^{c}|T(\mathfrak{a}) = \lambda(\mathfrak{a})f^{c}$ and $L(s,\mathbf{f},\chi) = L(s,\mathbf{f}^{c},\chi)$.*

3.2. **Doubling method.** The L-function introduced above may be also obtained via a doubling method. We chose to take Arakawa's approach [1] and considered a homomorphism

$$\iota_{A}\colon \boldsymbol{G}^{m,l} \times \boldsymbol{G}^{n,l} \to \boldsymbol{G}^{m+n,l},$$
$$\iota_{A}((\lambda,\mu,\kappa)g) \times (\lambda',\mu',\kappa')g')) := ((\lambda\,\lambda'),(\mu\,\mu'),\kappa+\kappa')\iota_{S}(g\times g'),$$

where

$$\iota_{S} : G^{m} \times G^{n} \hookrightarrow G^{m+n}, \quad \iota_{S}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \times \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)\right) := \begin{pmatrix} a & & b & \\ & a' & & b' \\ c & & d & \\ & c' & & d' \end{pmatrix}.$$

The map ι_{A} induces an embedding

$$\mathcal{H}_{m,l} \times \mathcal{H}_{n,l} \hookrightarrow \mathcal{H}_{n+m,l}, \ \ z_{1} \times z_{2} \mapsto \mathrm{diag}[z_{1},z_{2}],$$

defined by

$$(\tau_{1},w_{1}) \times (\tau_{2},w_{2}) \mapsto (\mathrm{diag}[\tau_{1},\tau_{2}],(w_{1}\,w_{2})).$$

The doubling method suggests that computation of the Petersson inner product of a cuspidal Siegel-Jacobi modular form f on $\mathcal{H}_{n,l}$ against a Siegel-type Eisenstein series pull-backed from $\mathcal{H}_{n+m,l}$ leads to an L-function associated with f. Before we state the result, we need to define an Eisenstein series.

Fix a weight $k \in \mathbb{Z}^{\mathbf{a}}$ and consider a Hecke character χ such that for a fixed integral ideal $\mathfrak{c}$ of F we have

(1) $\chi_v(x) = 1$ for all $x \in \mathfrak{o}_v^\times$ with $x - 1 \in \mathfrak{c}_v$, $v \in \mathbf{h}$,
(2) $\chi_{\mathbf{a}}(x_{\mathbf{a}}) = \mathrm{sgn}(x_{\mathbf{a}})^k := \prod_{v\in\mathbf{a}} \left(\frac{x_v}{|x_v|}\right)^{k_v}$, for $x_{\mathbf{a}} \in \mathbb{A}_{\mathbf{a}}$;

we will also write $\chi_{\mathfrak{c}} := \prod_{v|\mathfrak{c}} \chi_v$. We define an absolutely convergent adelic Eisenstein series of Siegel type on a Jacobi group with a parabolic subgroup

$$\boldsymbol{P}^n(F) := \{(0,\mu,\kappa)g : \mu \in M_{l,n}(F), \kappa \in Sym_l(F), g \in P^n(F)\},$$

where $P^n(F)$ is a Siegel subgroup of $G^n(F)$ as follows:

$$E(x,s;\chi) := \sum_{\gamma \in \boldsymbol{P}^n(F)\backslash \boldsymbol{G}^n(F)} \phi(\gamma x, s;\chi), \quad \mathrm{Re}(s) > \frac{1}{2}(n+l+1),$$

where $\phi(x,s;\chi) := 0$ if $x \notin \boldsymbol{P}^n(\mathbb{A})K^n$ and otherwise, if $x = \boldsymbol{pw}$ with $\boldsymbol{p} \in \boldsymbol{P}^n(\mathbb{A})$ and $\boldsymbol{w} \in K^n$, we set

$$\phi(x,s;\chi) := \chi(\det(d_p))^{-1}\chi_{\mathfrak{c}}(\det(d_w))^{-1} J_{k,S}(\boldsymbol{w},\mathbf{i}_0)^{-1}|\det(d_p)|_{\mathbb{A}}^{-2s},$$

where $p, w \in \mathrm{Sp}_n(\mathbb{A})$ denote symplectic parts of $\boldsymbol{p}, \boldsymbol{w}$, respectively. The classical Eisenstein series which corresponds to $E(x,s;\chi)$ via bijection (4) will be denoted by $E(z,s;\chi)$.

Theorem 3.3 ([3])**.** *Let $f \in S^n_{k,S}(\boldsymbol{\Gamma})$ be a Hecke eigenform and $E(z,s;\chi)$ an Eisenstein series defined above. Then:*

$$\begin{aligned} &G(\chi, 2s-n-l/2)N(\mathfrak{b})^{2ns}\chi_{\mathbf{h}}(\theta)^{-n}(-1)^{n(s-k/2)} vol(A)\Lambda^{2n}_{k-l/2,\mathfrak{c}}(s-l/4,\chi\psi_S)\\ &\quad\cdot < (E|_{k,S}\boldsymbol{\rho})(\mathrm{diag}[z_1,z_2],s;\chi), (f_{k,S}|\boldsymbol{\eta}_n)^c(z_2) > \\ &\qquad\qquad = \nu_{\mathfrak{c}} c_{S,k}(s-k/2)\boldsymbol{\Lambda}(s,f,\chi)f(z_1), \end{aligned} \tag{7}$$

where

$$\Lambda^{2n}_{k-l/2,\mathfrak{c}}(s,\chi\psi_S) := \begin{cases} L_{\mathfrak{c}}(2s-l/2,\chi\psi_S)\prod_{i=1}^n L_{\mathfrak{c}}(4s-l-2i,\chi^2) & \text{if } l \in 2\mathbb{Z},\\ \prod_{i=1}^{[(2n+1)/2]} L_{\mathfrak{c}}(4s-l-2i+1,\chi^2) & \text{if } l \notin 2\mathbb{Z},\end{cases}$$

$$\boldsymbol{\Lambda}(s,f,\chi) := L(2s-n-l/2,\mathbf{f},\chi)\begin{cases} L_{\mathfrak{c}}(2s-l/2,\chi\psi_S)\prod_{i=n+1}^n L_{\mathfrak{c}}(4s-l-2i,\chi^2), & l \in 2\mathbb{Z},\\ \prod_{i=n+1}^{[(2n+1)/2]} L_{\mathfrak{c}}(4s-l-2i+1,\chi^2), & l \notin 2\mathbb{Z},\end{cases}$$

$$G(\chi, 2s-n-l/2) := \prod_{(\mathfrak{p},\mathfrak{c})=1} G_{\mathfrak{p}}(\chi, 2s-n-l/2), \tag{8}$$

and the rest of notation is as in [3, Section 6]*; in particular: $\boldsymbol{\eta}_n = 1_H \begin{pmatrix} & -1_n \\ 1_n & \end{pmatrix}$,*

$$c_{S,k}(s) = \prod_{\nu\in\mathbf{a}}\left(\pm\det(2S_\nu)^{-n}2^{n(n+3)/2-4s_\nu-nk_\nu}\pi^{n(n+1)/2}\frac{\Gamma_n(s_\nu+k_\nu-\frac{l}{2}-\frac{n+1}{2})}{\Gamma_n(s_\nu+k_\nu-\frac{l}{2})}\right)$$

and $\Gamma_n(s) := \pi^{n(n-1)/4}\prod_{i=0}^{n-1}\Gamma(s-\frac{i}{2})$.

Statement of the above theorem expresses a combination of equations (30) and (31) from [3, Section 9] before multiplying them by the factor $\mathcal{G}_{k-l/2,2n}(s-l/4)$.

Remark 3.4. In fact, the results proved in [3] are more general than the ones presented above. Indeed, we worked with congruence subgroups of the form

$$\boldsymbol{D} := \{(\lambda,\mu,\kappa)x \in C_v[\mathfrak{o},\mathfrak{b}^{-1},\mathfrak{b}^{-1}]D_v[\mathfrak{b}^{-1}\mathfrak{e},\mathfrak{b}\mathfrak{c}] : (a_x - 1_n)_v \in M_n(\mathfrak{e}_v) \text{ for every } v|\mathfrak{e}\},$$

where $\mathfrak{e}$ is an integral ideal such that $\mathfrak{c} \subset \mathfrak{e}$ and $\mathfrak{e}$ is prime to $\mathfrak{e}^{-1}\mathfrak{c}$. If we set

$$\boldsymbol{\Gamma} := \boldsymbol{G}^n(F) \cap \boldsymbol{D},$$

then $\boldsymbol{\Gamma}_1(\mathfrak{c}) \subseteq \boldsymbol{\Gamma} \subseteq \boldsymbol{\Gamma}_0(\mathfrak{c})$ where the last group is obtained by setting $\mathfrak{e} = \mathfrak{o}$. However, in this paper we decided to work with $\mathfrak{e} = \mathfrak{c}$, because for simplicity reasons we restricted the proof of our main theorem to this case.

4. Arithmetic properties of Siegel-Jacobi modular forms

As we indicated in the introduction, assuming that one can define a sensible algebraic structure on the space of Siegel-Jacobi modular forms, it is natural to ask whether a "Deligne's Conjecture"-style result may hold for some values of the standard L-function, which are often called special L-values. This is indeed the case for Siegel modular forms, as shown for example in [13, 15]. Indeed, by using the theory of canonical models for the Siegel modular varieties (as it is explained in [13, Chapter 2]), one can define an algebraic structure on the space of Siegel modular forms, and for an algebraic eigenfunction establish algebraicity results for the special L-values of the attached standard L-function (see for example Theorem 28.8 in [13]). Furthermore, one can, conjecturally, attach a motive to such a Siegel modular form, such that the associated motivic L-function can be identified with the standard L-function (see for example [16]). Then the special values of the standard L-function can be identified with the critical values of the motivic L-function and then the algebraicity results can be seen in the light of Deligne's Period conjectures [4] (up to the difficult issue of comparing motivic and automorphic period).

It is then quite natural to ask whether the picture described above holds also for Siegel-Jacobi forms; that is, whether we can establish results towards the algebraicity of special L-values of Siegel-Jacobi modular forms. The starting point of our investigation is the paper of Shimura [11], where the arithmetic nature of Siegel-Jacobi modular forms is studied. We should remark right away that the paper of Shimura is written for $F = \mathbb{Q}$, but it is not very hard to see that almost everything there can be generalized to the situation of any totally real field F. Indeed, in what follows, whenever we state a result from that paper, we always comment on what is needed to extend it to the case of a totally real field.

In this section we change our convention: we will write $\boldsymbol{f}$ (instead of f) for Siegel-Jacobi modular forms, $\mathbf{f}$ will still denote the corresponding adelic form, and f will be used for other types of forms.

For a congruence subgroup $\boldsymbol{\Gamma}$ of $\boldsymbol{G}(F)$ as in the previous section and a subfield K of $\mathbb{C}$ we define the set

$$M^n_{k,S}(\boldsymbol{\Gamma}, K) := \{\boldsymbol{f} \in M^n_{k,S}(\boldsymbol{\Gamma}) : \boldsymbol{f}(\tau, w) = \sum_{t,r} c(t,r)\mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,(t\tau + {}^t r w)),\ c(t,r) \in K\};$$

the subspace $S^n_{k,S}(\boldsymbol{\Gamma}, K)$ consisting of cusp forms is defined in a similar way. Moreover, we write $M^n_{k,S}(K)$ for the union of all spaces $M^n_{k,S}(\boldsymbol{\Gamma}_1(\mathfrak{b},\mathfrak{c}), K)$ for all integral ideals $\mathfrak{c}$

and fractional ideals $\mathfrak{b}$, where $\boldsymbol{\Gamma}_1(\mathfrak{b},\mathfrak{c}) := \boldsymbol{G}^n(F) \cap \boldsymbol{D}_1(\mathfrak{b},\mathfrak{c})$, and

$\boldsymbol{D}_1(\mathfrak{b},\mathfrak{c}) := \{(\lambda,\mu,\kappa)x \in C[\mathfrak{o},\mathfrak{b}^{-1},\mathfrak{b}^{-1}]D[\mathfrak{b}^{-1}\mathfrak{c},\mathfrak{b}\mathfrak{c}] : (a_x - 1_n)_v \in M_n(\mathfrak{c}_v) \text{ for every } v|\mathfrak{c}\}$.

For an element $\sigma \in Aut(\mathbb{C})$ and an element $k = (k_v) \in \mathbb{Z}^{\mathbf{a}}$ we define $k^\sigma := (k_{v\sigma}) \in \mathbb{Z}^{\mathbf{a}}$, where $v\sigma$ is the archimedean place corresponding to the embedding $K \overset{\tau_v}{\hookrightarrow} \mathbb{C} \overset{\sigma}{\to} \mathbb{C}$, if τ_v is the embedding in $\mathbb{C}$ corresponding to the archimedean place v.

Proposition 4.1. *Let $k \in \mathbb{Z}^{\mathbf{a}}$, and let Φ be the Galois closure of F in $\overline{\mathbb{Q}}$, and Φ_k the subfield of Φ such that*

$$Gal(\Phi/\Phi_k) := \{\sigma \in Gal(\Phi/F) : \;\; k^\sigma = k\}.$$

Then $M^n_{k,S}(\mathbb{C}) = M^n_{k,S}(\Phi_k) \otimes_{\Phi_k} \mathbb{C}$.

Proof. If $F = \mathbb{Q}$, this is [11, Proposition 3.8]. A careful examination of the proof [11, page 60] shows that the proof is eventually reduced to the corresponding statement for Siegel modular forms of integral (if l is even) or half-integral (if l is odd) weight. However, in both cases the needed statement does generalize to the case of totally real fields, as it was established in [13, Theorems 10.4 and 10.7]. □

Given an $\boldsymbol{f} \in M^n_{k,S}(\mathbb{C})$, we define

$$\boldsymbol{f}_*(\tau, w) := \mathbf{e}_{\mathbf{a}}(Sw(\tau - \overline{\tau})^{-1}\,{}^t\!w)\boldsymbol{f}(\tau, w)$$

and write $\mathbb{Q}^{ab}$ for the maximal abelian extension of $\mathbb{Q}$. Moreover, for $k \in \frac{1}{2}\mathbb{Z}^{\mathbf{a}}$ such that $k_v - \frac{1}{2} \in \mathbb{Z}$ for all $v \in \mathbf{a}$ we write M^n_k for the space of Siegel modular forms of weight k, and of any congruence subgroup, and $M^n_k(K)$ for those with the property that all their Fourier coefficients at infinity lie in K (see for example [13, Chapter 2] for a detailed study of these sets).

Proposition 4.2. *Let K be a field that contains $\mathbb{Q}^{ab}$ and Φ as above. Then*

(1) *$\boldsymbol{f} \in M^n_{k,S}(K)$ if and only if $\boldsymbol{f}_*(\tau, v\Omega_\tau) \in M^n_k(K)$, where $\Omega_\tau := {}^t(\tau \;\, 1_n)$, and $v \in M_{l,2n}(F)$.*
(2) *For any element $\gamma \in \mathrm{Sp}_n(F) \hookrightarrow \boldsymbol{G}^n(F)$ and $\boldsymbol{f} \in M^n_{k,S}(K)$, we have*

$$\boldsymbol{f}|_{k,S}\gamma \in M^n_{k,S}(K).$$

Moreover if $\boldsymbol{f} \in M^n_{k,S}(\boldsymbol{\Gamma}, K)$, it follows that $\boldsymbol{f}|T_r \in M^n_{k,S}(\boldsymbol{\Gamma}, K)$ for any $r \in Q(\mathfrak{c})$.

Proof. If $F = \mathbb{Q}$, this is [11, Proposition 3.2]. It is easy to see that the proof generalizes to the case of any totally real field. Indeed, the first part of the proof is a direct generalization of the argument used by Shimura. The second part requires the fact that the space $M^n_k(K)$ is stable under the action of elements in $\mathrm{Sp}_n(F)$, which is true for any totally real field, as it is proved in [13, Theorem 10.7 (6)]. The last statement follows from the definition of the Hecke operator $T_{r,\psi}$. □

For a symmetric matrix $S \in Sym_l(F)$, $h \in M_{l,n}(F)$ and a lattice $L \subset M_{l,n}(F)$ we define the Jacobi theta series of characteristic h by

$$\Theta_{S,L,h}(\tau, w) = \sum_{x \in L} \mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,(S(\frac{1}{2}{}^t(x+h)\tau(x+h) + (x+h)w))).$$

Theorem 4.3. *Assume that $n > 1$ or $F \neq \mathbb{Q}$, and let K be any subfield of $\mathbb{C}$. Let $A \in \mathrm{GL}_l(F)$ be such that $AS\,{}^tA = \mathrm{diag}[s_1, \ldots, s_l]$, and define the lattices $\Lambda_1 := AM_{l,n}(\mathfrak{o}) \subset M_{l,n}(F)$ and $\Lambda_2 := 2\mathrm{diag}[s_1^{-1}, \ldots, s_l^{-1}]M_{l,n}(\mathfrak{o}) \subset M_{l,n}(F)$. Then there is an isomorphism*

$$\Phi : M^n_{k,S}(K) \cong \bigoplus_{h \in \Lambda_1/\Lambda_2} M^n_{k-l/2}(K)$$

given by $\boldsymbol{f} \mapsto (f_h)_h$, where the $f_h \in M^n_{k-l/2}(K)$ are defined by the expression

$$\boldsymbol{f}(\tau, w) = \sum_{h \in \Lambda_1/\Lambda_2} f_h(\tau)\Theta_{2S,\Lambda_2,h}(\tau, w).$$

Moreover, under the above isomorphism,

$$\Phi^{-1}\left(\bigoplus_{h \in \Lambda_1/\Lambda_2} S^n_{k-l/2}(K)\right) \subset S^n_{k,S}(K).$$

Remark 4.4. We remark here that the assumption of $n > 1$ or $F \neq \mathbb{Q}$ is needed to guarantee that the f_h's are holomorphic at the cusps, which follows from the Köcher principle. However, even in the case of $F = \mathbb{Q}$ and $n = 1$, if we take $\boldsymbol{f}$ to be of trivial level, then the f_h's are holomorphic at infinity (see for example [5, page 59]).

Proof of Theorem 4.3. The first statement is [11, Proposition 3.5] for $F = \mathbb{Q}$ and it easily generalizes to the case of any totally real field. We explain the statement about cusp forms.

Consider first expansions around the cusp at infinity. Fix $h \in \Lambda_1/\Lambda_2$ and let $f_h(\tau) = \sum_{t_2>0} c(t_2)\mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,(t_2\tau))$. It is known that Fourier coefficients $c(t_1, r)$ of a Jacobi theta series

$$\Theta_{2S,\Lambda_2,h}(\tau, w) = \sum_{t_1,r} c(t_1, r)\mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,(t_1\tau))\mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,({}^t r w))$$

are nonzero only if $4t_1 = rS^{-1}\,{}^t r$ (see [17, p. 210]). Hence, the coefficients of

$$f_h(\tau)\Theta_{2S,\Lambda_2,h}(\tau, w) = \sum_{t,r}\left(\sum_{t_1+t_2=t} c(t_1, r)c(t_2)\right)\mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,(t\tau))\mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,({}^t r w))$$

are nonzero only if $4t = 4(t_1 + t_2) = rS^{-1}\,{}^t r + 4t_2 > rS^{-1}\,{}^t r$. This means that the function $f_h(\tau)\Theta_{2S,\Lambda_2,h}(\tau, w)$ satisfies cuspidality condition at infinity.

Now let γ be any element in $\mathrm{Sp}_n(F)$. The first statement in the Theorem states that for every $h_1 \in \Lambda_1/\Lambda_2$ there exist $f_{h_1,h_2} \in M^n_{k-l/2}(K), h_2 \in \Lambda_1/\Lambda_2$, such that

$$\Theta_{2S,\Lambda_2,h_1}|_{k,S}\gamma(\tau, w) = \sum_{h_2} f_{h_1,h_2}(\tau)\Theta_{2S,\Lambda_2,h_2}(\tau, w).$$

Hence, for some cusp forms $f_{h_1} \in S^n_{k-l/2}(K)$,

$$\boldsymbol{f}|_{k,S}\gamma(\tau, w) := \sum_{h_1} f_{h_1}|_k\gamma(\tau)\left(\sum_{h_2} f_{h_1,h_2}(\tau)\Theta_{2S,\Lambda_2,h_2}(\tau, w)\right)$$

$$= \sum_{h_2}\left(\sum_{h_1} f_{h_1}|_k\gamma(\tau) f_{h_1,h_2}(\tau)\right)\Theta_{2S,\Lambda_2,h_2}(\tau,w).$$

The same argument as used for the cusp at infinity implies that the functions $\boldsymbol{f}|_{k,S}\gamma(\tau,w)$ and $\sum_{h_1} f_{h_1}|_k\gamma(\tau)f_{h_1,h_2}(\tau)$ are cuspidal. This finishes the proof. $\square$

Note that the above theorem does not state that $\Phi^{-1}\left(\bigoplus_{h\in\Lambda_1/\Lambda_2} S^n_{k-l/2}(K)\right) = S^n_{k,S}(K)$. For this reason we make the following definition.

Property A. We say that a cusp form $\boldsymbol{f} \in S^n_{k,S}(K)$ has the Property A if

$$\Phi(\boldsymbol{f}) \in \bigoplus_{h\in\Lambda_1/\Lambda_2} S^n_{k-l/2}(K).$$

Examples of Siegel-Jacobi forms that satisfy the Property A:

(1) Siegel-Jacobi forms over a field F of class number one, and with trivial level, i.e. with $\mathfrak{c} = \mathfrak{o}$. Note that in this situation there is only one cusp. Then, keeping the notation as in the proof of the theorem above we need to verify that if $\boldsymbol{f}(\tau,w) = \sum_{t,r} c_{\mathbf{f}}(t,r)\mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,(t\tau))\mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,({}^t\!rw))$ with $4t > rS^{-1}\,{}^t\!r$ whenever $c(t,r) \neq 0$, then the f_h have to be cuspidal. Observe first that if $h_1,h_2 \in \Lambda_1/\Lambda_2$ are different, $\Theta_{2S,\Lambda_2,h_1}(\tau,w) = \sum_{t,r} c_1(t,r)\mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,(t\tau))\mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,({}^t\!rw))$, and $\Theta_{2S,\Lambda_2,h_2}(\tau,w) = \sum_{t,r} c_2(t,r)\mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,(t\tau))\mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,({}^t\!rw))$, then there is no r such that at the same time $c_1(t,r) \neq 0$ and $c_2(t,r) \neq 0$. Indeed, if it was not the case then there would be $\lambda_1,\lambda_2 \in \Lambda_2$ such that ${}^t\!r = 2S(\lambda_1+h_1)$ and ${}^t\!r = 2S(\lambda_2+h_2)$, that is, $\lambda_1+h_1 = \lambda_2+h_2$ or, equivalently, $h_1-h_2 \in \Lambda_2$; contradiction. Hence, for any given r there is a unique $h \in \Lambda_1/\Lambda_2$ such that $\Theta_{2S,\Lambda_2,h}$ has a nonzero coefficient $c(t,r)$. This means that there exists a unique h such that $c_{\mathbf{f}}(t,r)$ is the Fourier coefficient of $f_h(\tau)\Theta_{2S,\Lambda_2,h}(\tau,w) = \sum_{t,r}\sum_{t_1+t_2=t} c(t_1,r)c(t_2)\mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,(t\tau))\mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,({}^t\!rw))$. But then $rS^{-1}\,{}^t\!r < 4t = 4(t_1+t_2) = rS^{-1}\,{}^t\!r + 4t_2$ and so $t_2 > 0$, which proves that f_h is cuspidal.

(2) Siegel-Jacobi forms of index S such that $\det(2S) \in \mathfrak{o}^\times$, as in this case the lattices Λ_1 and Λ_2 from Theorem 4.3 are equal.

(3) Siegel-Jacobi forms of non-parallel weight, that is, if there exist distinct $v, v' \in \mathbf{a}$ such that $k_v \neq k_{v'}$. Indeed, in this case $M^n_{k-l/2}(K) = S^n_{k-l/2}(K)$ for all $h \in \Lambda_1/\Lambda_2$ (see [12, Proposition 10.6]).

Let us now explain the significance of the Property A. Recall first that we have defined a Petersson inner product $<\boldsymbol{f},\boldsymbol{g}>$ when $\boldsymbol{f},\boldsymbol{g} \in M^n_{k,S}(K)$ and one of them, say, $\boldsymbol{f}$ is cuspidal. If $\boldsymbol{f}$ satisfies the Property A, then we claim that

$$<\boldsymbol{f},\boldsymbol{g}>= N(\det(4S))^{-n/2}\sum_{h\in\Lambda_1/\Lambda_2} <f_h,g_h>.$$

Indeed, as in [17, Lemma 3.4],

$$<\boldsymbol{f},\boldsymbol{g}>= N(\det(4S))^{-n/2}vol(A)^{-1}\int_A \sum_{h\in\Lambda_1/\Lambda_2} f_h(\tau)\overline{g_h(\tau)}\det(\mathrm{Im}(\tau))^{k-l/2-(n+1)}d\tau,$$

where $A = \Gamma\backslash\mathbb{H}_n^{\mathbf{a}}$ and a congruence subgroup Γ is deep enough. We obtain the claimed equality after exchanging the order of integration and summation. This can be done exactly because each f_h is cuspidal, which makes each individual integral well defined.

Lemma 4.5. *Assume that $n > 1$ or $F \neq \mathbb{Q}$ and that $\boldsymbol{f} \in S^n_{k,S}(\overline{\mathbb{Q}})$ satisfies the Property A and one of the following two conditions holds:*

(i) there exist $v, v' \in \mathbf{a}$ such that $k_v \neq k_{v'}$;
(ii) $k = \mu\mathbf{a} = (\mu, \dots, \mu) \in \mathbb{Z}^{\mathbf{a}}$, with $\mu \in \mathbb{Z}$ depending on n and F in the following way:

$$\begin{array}{cccc} n > 2 & n = 2, F = \mathbb{Q} & n = 2, F \neq \mathbb{Q} & n = 1 \\ \mu > 3n/2 + l/2 & \mu > 3 & \mu > 2 & \mu \geq 1/2 \end{array}.$$

Then for any $\boldsymbol{g} \in M^n_{k,S}(\overline{\mathbb{Q}})$ there exists $\widetilde{\boldsymbol{g}} := \mathfrak{q}(\boldsymbol{g}) \in S^n_{k,S}(\overline{\mathbb{Q}})$ such that

$$< \boldsymbol{f}, \boldsymbol{g} > = < \boldsymbol{f}, \widetilde{\boldsymbol{g}} > .$$

Proof. There is nothing to show in the case of non-parallel weight, since as it was mentioned above there is no (holomorphic) Eisenstein part in this case. In the parallel weight case, since $\boldsymbol{f}$ has the Property A, $< \boldsymbol{f}, \boldsymbol{g} > = N(\det(4S))^{-n/2} \sum_{h \in \Lambda_1/\Lambda_2} < f_h, g_h >$. Let $\widetilde{\mathfrak{q}} : M^n_{k-l/2}(\overline{\mathbb{Q}}) \to S^n_{k-l/2}(\overline{\mathbb{Q}})$ be the projection operator defined in [13, Theorem 27.14]. Then, if we put $\widetilde{g}_h := \widetilde{\mathfrak{q}}(g_h)$ for all $h \in \Lambda_1/\Lambda_2$, it follows that

$$< \boldsymbol{f}, \boldsymbol{g} > = N(\det(4S))^{-n/2} \sum_{h \in \Lambda_1/\Lambda_2} < f_h, g_h > = N(\det(4S))^{-n/2} \sum_{h \in \Lambda_1/\Lambda_2} < f_h, \widetilde{g}_h > .$$

In particular, if we set $\widetilde{\boldsymbol{g}} := \Phi^{-1}((\widetilde{g}_h)_h)$, we obtain the statement of the lemma. $\square$

We consider now a non-zero $\boldsymbol{f} \in S^n_{k,S}(\boldsymbol{\Gamma}, \overline{\mathbb{Q}})$ with $\boldsymbol{\Gamma} := \boldsymbol{G} \cap \boldsymbol{D}$, where

$\boldsymbol{D} := \{(\lambda, \mu, \kappa)x \in C[\mathfrak{o}, \mathfrak{b}^{-1}, \mathfrak{b}^{-1}]D[\mathfrak{b}^{-1}\mathfrak{c}_{\boldsymbol{f}}, \mathfrak{b}\mathfrak{c}_{\boldsymbol{f}}] : (a_x - 1_n)_v \in M_n((\mathfrak{c}_{\boldsymbol{f}})_v) \text{ for every } v | \mathfrak{c}_{\boldsymbol{f}}\}$.

We assume that $\boldsymbol{f}$ is an eigenfunction of the operators $T(\mathfrak{a})$ for all integral ideals $\mathfrak{a}$, write $\boldsymbol{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\boldsymbol{f}$ and define the space

$$V(\boldsymbol{f}) := \{\widetilde{\boldsymbol{f}} \in S^n_{k,S}(\boldsymbol{\Gamma}, \overline{\mathbb{Q}}) : \widetilde{\boldsymbol{f}}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\widetilde{\boldsymbol{f}} \text{ for all } \mathfrak{a}\}.$$

We are now ready to state the main theorem of this paper on algebraic properties of

$$\boldsymbol{\Lambda}(s, \boldsymbol{f}, \chi) = L(2s - n - l/2, \boldsymbol{f}, \chi) \begin{cases} L_{\mathfrak{c}}(2s - l/2, \chi\psi_S) & \text{if } l \in 2\mathbb{Z}, \\ 1 & \text{if } l \notin 2\mathbb{Z}. \end{cases}$$

Theorem 4.6. *Assume $n > 1$ or $F \neq \mathbb{Q}$. Let χ be a Hecke character of F such that $\chi_{\boldsymbol{a}}(x) = \operatorname{sgn}_{\boldsymbol{a}}(x)^k$, and $0 \neq \boldsymbol{f} \in S^n_{k,S}(\boldsymbol{\Gamma}, \overline{\mathbb{Q}})$ an eigenfunction of all $T(\mathfrak{a})$. Set $\mu := \min_v k_v$ and assume that*

(1) $\mu > 2n + l + 1$,
(2) Property A holds for all $\widetilde{\boldsymbol{f}} \in V(\boldsymbol{f})$,
(3) $k_v \equiv k_{v'} \mod 2$ for all $v, v' \in \mathbf{a}$.

Let $\sigma \in \mathbb{Z}$ be such that

(1) $2n + 1 - (k_v - l/2) \leq \sigma - l/2 \leq k_v - l/2$ for all $v \in \mathbf{a}$,
(2) $|\sigma - \frac{l}{2} - \frac{2n+1}{2}| + \frac{2n+1}{2} - (k_v - l/2) \in 2\mathbb{Z}$ for all $v \in \mathbf{a}$,

(3) $k_v > l/2 + n(1 + k_v - l/2 - |\sigma - l/2 - (2n+1)/2| - (2n+1)/2)$ *for all* $v \in \mathbf{a}$,

but exclude the cases

(1) $\sigma = n + 1 + l/2$, $F = \mathbb{Q}$ *and* $\chi^2\psi_i^2 = 1$ *for some* ψ_i,
(2) $\sigma = l/2$, $\mathfrak{c} = \mathfrak{o}$ *and* $\chi\psi_S\psi_i = 1$ *for some* ψ_i,
(3) $0 < \sigma - l/2 \leq n$, $\mathfrak{c} = \mathfrak{o}$ *and* $\chi^2\psi_i^2 = 1$ *for some* ψ_i.
(4) $\sigma \leq l + n$ *in case* F *has class number larger than one.*

Under these conditions

$$\frac{\mathbf{\Lambda}(\sigma/2, \boldsymbol{f}, \chi)}{\pi^{e_\sigma} < \boldsymbol{f}, \boldsymbol{f} >} \in \overline{\mathbb{Q}},$$

where

$$e_\sigma = n \sum_{v \in \mathbf{a}} (k_v - l + \sigma) - de, \quad e := \begin{cases} n^2 + n - \sigma + l/2, & \text{if } 2\sigma - l \in 2\mathbb{Z} \text{ and } \sigma \geq 2n + l/2, \\ n^2, & \text{otherwise.} \end{cases}$$

This theorem will be proved at the end of the next section. First we need to introduce the notion of nearly holomorphic Siegel-Jacobi modular forms $N_{k,S}^{n,r}(\mathbf{\Gamma})$ for $r \in \mathbb{Z}^{\mathbf{a}}$.

5. Nearly holomorphic Siegel-Jacobi modular forms and algebraicity of special L-values

Definition 5.1. A C^∞ function $\boldsymbol{f}(\tau, w) : \mathcal{H}_{n,l} \to \mathbb{C}$ is said to be a nearly holomorphic Siegel-Jacobi modular form (of weight k and index S) for the congruence subgroup $\mathbf{\Gamma}$ if

(1) $\boldsymbol{f}$ is holomorphic with respect to the variable w and nearly holomorphic with respect to the variable τ, that is, $\boldsymbol{f}$ belongs to the space $N^r(\mathbb{H}_n^d)$ for some $r \in \mathbb{N}$ defined in [13, page 99];
(2) $\boldsymbol{f}|_{k,S}\gamma = \boldsymbol{f}$ for all $\gamma \in \mathbf{\Gamma}$.

We denote this space by $N_{k,S}^{n,r}(\mathbf{\Gamma})$ and write $N_{k,S}^{n,r} := \bigcup_{\mathbf{\Gamma}} N_{k,S}^{n,r}(\mathbf{\Gamma})$ for the space of all nearly holomorphic Siegel-Jacobi modular forms of weight k and index S.

We note that if $\boldsymbol{f} \in N_{k,S}^{n,r}$, then $\boldsymbol{f}_*(\tau, v\,\Omega_\tau) \in N_k^{n,r}$, the space of nearly holomorphic Siegel modular forms, where recall $\Omega_\tau := {}^t(\tau\ \ 1_n)$, and $v \in M_{l,2n}(F)$. Below we extend Theorem 4.3 to the nearly-holomorphic situaton.

Theorem 5.2. *Assume that* $n > 1$ *or* $F \neq \mathbb{Q}$. *Let* $A \in \mathrm{GL}_l(F)$ *be such that* $AS\,{}^tA = \mathrm{diag}[s_1, \ldots, s_l]$, *and define the lattices* $\Lambda_1 := AM_{l,n}(\mathfrak{o}) \subset M_{l,n}(F)$ *and* $\Lambda_2 := 2\mathrm{diag}[s_1^{-1}, \ldots, s_l^{-1}]M_{l,n}(\mathfrak{o}) \subset M_{l,n}(F)$. *Then there is an isomorphism*

$$\Phi : N_{k,S}^{n,r} \cong \bigoplus_{h \in \Lambda_1/\Lambda_2} N_{k-l/2}^{n,r}$$

given by $\boldsymbol{f} \mapsto (f_h)_h$, *where the* $f_h \in N_{k-l/2}^{n,r}$ *are defined by the expression*

$$\boldsymbol{f}(\tau, w) = \sum_{h \in \Lambda_1/\Lambda_2} f_h(\tau)\Theta_{2S,\Lambda_2,h}(\tau, w).$$

Proof. Given an $\boldsymbol{f} \in N^{n,r}_{k,S}$, the modularity properties with respect to the variable w show that (see for example [11, proof of Proposition 3.5]) we may write

$$\boldsymbol{f}(\tau, w) = \sum_{h \in \Lambda_1/\Lambda_2} f_h(\tau)\Theta_{2S,\Lambda_2,h}(\tau, w)$$

for some functions $f_h(\tau)$ with the needed modularity properties. In order to establish that they are actually nearly holomorphic one argues similarly to the holomorphic case. Indeed, a close look at the proof of [11, Lemma 3.4] shows that the functions f_h have the same properties (real analytic, holomorphic, nearly holomorphic, meromorphic, etc.) with respect to the variable τ as $\boldsymbol{f}(\tau, w)$, since everything is reduced to a linear system of the form

$$\boldsymbol{f}(\tau, w_i) = \sum_{h \in \Lambda_1/\Lambda_2} f_h(\tau)\Theta_{2S,\Lambda_2,h}(\tau, w_i), \ \ i = 1, \dots, \sharp\Lambda_1/\Lambda_2,$$

for some $\{w_i\}$ such that $\det(\Theta_{2S,\Lambda_2,h}(\tau, w_i)) \neq 0$. In particular, after solving the linear system of equations we see that the nearly holomorphicity of f_h follows from that of $\boldsymbol{f}$ since the $\Theta_{2S,\Lambda_2,h}(\tau, w_i)$ are holomorphic with respect to the variable τ. □

The above theorem immediately implies the following.

Corollary 5.3. *For a congruence subgroup $\boldsymbol{\Gamma}$, $N^{n,r}_{k,S}(\boldsymbol{\Gamma})$ is a finite dimensional $\mathbb{C}$ vector space.*

Proof. The theorem above states that $N^{n,r}_{k,S}(\boldsymbol{\Gamma}) \cong \bigoplus_h N^{n,r}_{k-l/2}(\Gamma_h)$ for some congruence subgroups Γ_h, which are known to be finite dimensional (see [13, Lemma 14.3]). □

Given an automorphism $\sigma \in Aut(\mathbb{C})$ and $\boldsymbol{f} \in N^{n,r}_{k,S}$, we define

$$\boldsymbol{f}^\sigma(\tau, w) := \sum_{h \in \Lambda_1/\Lambda_2} f^\sigma_h(\tau)\Theta_{2S,\Lambda_2,h}(\tau, w),$$

where $f_h \in N^{n,r}_{k-l/2}$, and f^σ_h is defined as in [13, page 117]. Also, for a subfield K of $\mathbb{C}$, define the space $N^{n,r}_{k,S}(K)$ to be the subspace of $N^{n,r}_{k,S}$ such that $\Phi(N^{n,r}_{k,S}(K)) = \bigoplus_{h \in \Lambda_1/\Lambda_2} N^n_{k-l/2}(K)$. In particular, $\boldsymbol{f} \in N^{n,r}_{k,S}$ belongs to $N^{n,r}_{k,S}(K)$ if and only if $\boldsymbol{f}^\sigma = \boldsymbol{f}$ for all $\sigma \in Aut(\mathbb{C}/K)$. Moreover, if K contains the Galois closure of F in $\overline{\mathbb{Q}}$ and $\mathbb{Q}^{ab}$, then $N^{n,r}_{k,S} = N^{n,r}_{k,S}(K) \otimes_K \mathbb{C}$ as the same statement holds for $N^{n,r}_{k-l/2}$. Similarly it follows that if $\boldsymbol{f} \in N^{n,r}_{k,S}(\overline{\mathbb{Q}})$, then $\boldsymbol{f}|_{k,S}\boldsymbol{\gamma} \in N^{n,r}_{k,S}(\overline{\mathbb{Q}})$ for all $\boldsymbol{\gamma} \in \boldsymbol{G}(F)$. At this point we also remark that for an $\boldsymbol{f} \in M^n_{k,S}$ the $\boldsymbol{f}^c$ defined before is nothing else than $\boldsymbol{f}^\rho$ where $1 \neq \rho \in Gal(\mathbb{C}/\mathbb{R})$ i.e. a complex conjugation.

We now define a variant of the holomorphic projection in the Siegel-Jacobi case. We define a map $\mathfrak{p}\colon N^{n,r}_{k,S}(\overline{\mathbb{Q}}) \to M^n_{k,S}(\overline{\mathbb{Q}})$ whenever $k_v > n + r_v$ for all $v \in \mathbf{a}$ by

$$\mathfrak{p}(\boldsymbol{f}) := \mathfrak{p}\left(\sum_{h \in \Lambda_1/\Lambda_2} f_h(\tau)\Theta_{2S,\Lambda_2,h}(\tau, w)\right) := \sum_{h \in \Lambda_1/\Lambda_2} \widetilde{\mathfrak{p}}(f_h(\tau))\Theta_{2S,\Lambda_2,h}(\tau, w),$$

where $\widetilde{\mathfrak{p}}\colon N^{n,r}_{k-l/2}(\overline{\mathbb{Q}}) \to M^n_{k-l/2}(\overline{\mathbb{Q}})$ is the holomorphic projection operator defined for example in [13, Chapter III, section 15].

Lemma 5.4. *Assume $n > 1$ or $F \neq \mathbb{Q}$ and that $\boldsymbol{f} \in S^n_{k,S}$ satisfies the Property A, and $k_v > n + r_v$ for all $v \in \mathbf{a}$. Then for any $\boldsymbol{g} \in N^{n,r}_{k,S}(\overline{\mathbb{Q}})$,*

$$< \boldsymbol{f}, \boldsymbol{g} >=< \boldsymbol{f}, \mathfrak{p}(\boldsymbol{g}) > .$$

Proof. This follows from the fact that the above property holds for nearly holomorphic Siegel modular forms, and the fact that the Property A allows us to write the Petersson inner product of Siegel-Jacobi forms as a sum of Petersson inner products of Siegel modular forms, in a similar way as we did in the proof of Lemma 4.5. □

Let us write F_1 for the Hilbert class field extension of F and denote by $\{\psi_i\}$ the ideal characters corresponding to the characters of $Gal(F_1/F)$. We can now state a theorem regarding the nearly holomorphicity of Siegel-type Jacobi Eisenstein series.

Theorem 5.5. *Consider the normalized Siegel-type Jacobi-Eisenstein series*

$$D(s) := D(z, s; k, \chi) := \Lambda^n_{k-l/2,\mathfrak{c}}(s - l/4, \chi\psi_S)E(z, \chi, s).$$

Let $\mu \in \mathbb{Z}$ be such that

(1) $n + 1 - (k_v - l/2) \leq \mu - l/2 \leq k_v - l/2$ for all $v \in \mathbf{a}$, and
(2) $|\mu - l/2 - \frac{n+1}{2}| + \frac{n+1}{2} - k_v + l/2 \in 2\mathbb{Z}$,

but exclude the cases

(1) $\mu = \frac{n+2}{2} + l/2$, $F = \mathbb{Q}$ and $\chi^2\psi_i^2 = 1$ for some ψ_i,
(2) $\mu = l/2$, $\mathfrak{c} = \mathfrak{o}$ and $\chi\psi_S\psi_i = 1$ for some ψ_i,
(3) $0 < \mu - l/2 \leq n/2$, $\mathfrak{c} = \mathfrak{o}$ and $\chi^2\psi_i^2 = 1$ for some ψ_i.
(4) $\mu \leq l + n$ if F has class number larger than one.

Then

$$D(\mu/2) \in \pi^\beta N^{n,r}_{k,S}(\overline{\mathbb{Q}}),$$

where

$$r = \begin{cases} \frac{n(k-\mu+2)}{2} & \text{if } \mu = \frac{n+2}{2} + \frac{l}{2},\ F = \mathbb{Q}, \chi^2 = 1, \\ \frac{k}{2} - \frac{l}{4} & \text{if } n = 1, \mu = 2 + \frac{l}{2}, F = \mathbb{Q}, \chi\psi_S = 1, \\ \frac{n}{2}(k - \frac{l}{2} - |\mu - \frac{l}{2} - \frac{n+1}{2}|\mathbf{a} - \frac{n+1}{2}\mathbf{a}) & \text{otherwise.} \end{cases}$$

Moreover, $\beta = \frac{n}{2}\sum_{v\in\mathbf{a}}(k_v - l + \mu) - de$, where

$$e := \begin{cases} [\frac{(n+1)^2}{4}] - \mu + \frac{l}{2} & \text{if } 2\mu - l + n \in 2\mathbb{Z}, \mu \geq n + \frac{l}{2}, \\ [\frac{n^2}{4}] & \text{otherwise.} \end{cases}$$

Proof. The proof is similar to the proof of Theorem 8.3 in [3], where the analytic properties of this series were established. As in there, we can read off the nearly holomorphicity of the Jacobi Eisenstein series from the classical Siegel Eisenstein series, which are given in [13, Theorem 17.9]; to be more precise, from the Siegel-type series

$E(\tau, s-l/4; \chi\psi_S\psi_i, k-l/2)$ (with the notation as in [3]), where ψ_i's vary over all the Hilbert characters. Indeed, the series

$$\frac{\Lambda^n_{k-l/2,\mathfrak{c}}(\mu/2-l/4,\chi\psi_S)}{\Lambda^n_{k-l/2,\mathfrak{c}}(\mu/2-l/4,\chi\psi_S\psi_i)}\Lambda^n_{k-l/2,\mathfrak{c}}(\mu/2-l/4,\chi\psi_S\psi_i)E(\tau,s-l/4;\chi\psi_S\psi_i,k-l/2)$$

has the same algebraic properties as the normalized series

$$\Lambda^n_{k-l/2,\mathfrak{c}}(\mu/2-l/4,\chi\psi_S\psi_i)E(\tau,s-l/4;\chi\psi_S\psi_i,k-l/2),$$

if we exclude the cases where the factor $\frac{\Lambda^n_{k-l/2,\mathfrak{c}}(\mu/2-l/4,\chi\psi_S)}{\Lambda^n_{k-l/2,\mathfrak{c}}(\mu/2-l/4,\chi\psi_S\psi_i)}$ has a pole. Therefore all we need to check is that

$$\frac{\Lambda^n_{k-l/2}(\mu/2-l/4,\chi\psi_S)}{\Lambda^n_{k-l/2}(\mu/2-l/4,\chi\psi_S\psi_i)} \in \overline{\mathbb{Q}}.$$

This should follow from the general Bellinson conjectures for motives associated to finite Hecke characters over totally real fields (see for example [10]). However this is not known in general, and hence we are forced to set the condition $\mu > n+l$ in case F has class number larger than one, in which case we obtain values whose ratio is known to be algebraic, since we are then considering critical values. $\square$

Lemma 5.6. *Consider the embedding*

$$\Delta : \mathcal{H}_{n,l} \times \mathcal{H}_{m,l} \hookrightarrow \mathcal{H}_{n+m,l}, \quad (\tau_1, w_1) \times (\tau_2, w_2) \mapsto (\mathrm{diag}[\tau_1,\tau_2], (w_1\ w_2)).$$

Then the pullback

$$\Delta^*\left(N^{n+m,r}_{k,S}(\overline{\mathbb{Q}})\right) \subset N^{n,r}_{k,S}(\overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} N^{m,r}_{k,S}(\overline{\mathbb{Q}}).$$

Proof. The proof of this lemma is identical to the Siegel modular form case (see [13, Lemma 24.11]). Let $\boldsymbol{f} \in N^{n+m,r}_{k,S}(\boldsymbol{\Gamma}^{n+m}, \overline{\mathbb{Q}})$ for a sufficiently deep congruence subgroup $\boldsymbol{\Gamma}^{n+m}$. Note that the function $\boldsymbol{g}(z_1,z_2) := \Delta^*\boldsymbol{f}(\mathrm{diag}[z_1,z_2])$ is in $N^{n,r}_{k,S}(\boldsymbol{\Gamma}^n)$ as a function in z_1 and in $N^{m,r}_{k,S}(\boldsymbol{\Gamma}^m)$ as a function in z_2 for appropriate congruence subgroups $\boldsymbol{\Gamma}^n$ and $\boldsymbol{\Gamma}^m$. Hence, by Corollary 5.3 and the fact that $N^{n,r}_{k,S} = N^{n,r}_{k,S}(\overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$, for each fixed z_1 we may write

$$\boldsymbol{g}(z_1,z_2) = \sum_i \boldsymbol{g}_i(z_1)\boldsymbol{h}_i(z_2),$$

where $\boldsymbol{g}_i(z_1) \in \mathbb{C}$, and $\boldsymbol{h}_i \in N^{n,r}_{k,S}(\overline{\mathbb{Q}})$ form a basis of the space. The general argument used in [13, Lemma 24.11], which is based on the linear independence of the basis $\boldsymbol{h}_i$, shows that the functions $\boldsymbol{g}_i(z_1)$ have the same properties as the function $\boldsymbol{g}$ when viewed as a function of the variable z_1. Hence, $\boldsymbol{g}_i \in N^{n,r}_{k,S}$. Now, for any $\sigma \in Aut(\mathbb{C}/\overline{\mathbb{Q}})$,

$$\boldsymbol{g}(z_1,z_2) = \boldsymbol{g}^\sigma(z_1,z_2) = \sum_i \boldsymbol{g}_i^\sigma(z_1)\boldsymbol{h}_i^\sigma(z_2) = \sum_i \boldsymbol{g}_i^\sigma(z_1)\boldsymbol{h}_i(z_2).$$

Hence, $\boldsymbol{g}_i^\sigma(z_2) = \boldsymbol{g}_i(z_2)$ for all $\sigma \in Aut(\mathbb{C}/\overline{\mathbb{Q}})$, and thus $\boldsymbol{g}_i \in N^{n,r}_{k,S}(\overline{\mathbb{Q}})$. $\square$

We can now establish a theorem which is the key result towards Theorem 4.6.

Theorem 5.7. *Assume $n > 1$ or $F \neq \mathbb{Q}$. Let $0 \neq \boldsymbol{f} \in S^n_{k,S}(\boldsymbol{\Gamma}, \overline{\mathbb{Q}})$ be an eigenfunction of $T(\mathfrak{a})$ for all integral ideas $\mathfrak{a}$ with $(\mathfrak{a}, \mathfrak{c}_{\boldsymbol{f}}) = 1$. Define $\mu := \min_{v \in \mathbf{a}} \{k_v\}$ and assume that*

(1) $\mu > 2n + l + 1$,
(2) Property A holds for all $\widetilde{\boldsymbol{f}} \in V(\boldsymbol{f})$,
(3) $k_v \equiv k_{v'} \mod 2$ for all $v, v' \in \mathbf{a}$.
(4) $k_v > l/2 + n(1 + k_v - \mu)$ for all $v \in \mathbf{a}$.

Then for any $\boldsymbol{g} \in M^n_{k,S}(\overline{\mathbb{Q}})$,

$$\frac{< \boldsymbol{f}, \boldsymbol{g} >}{< \boldsymbol{f}, \boldsymbol{f} >} \in \overline{\mathbb{Q}}$$

Proof. By Lemma 4.5 it suffices to prove this theorem for $\boldsymbol{g} \in S^n_{k,S}(\overline{\mathbb{Q}})$. Furthermore, as it was shown in [3, section 7.4], the Hecke operators are normal and Proposition 4.2 states that the Hecke operators $T(\mathfrak{a})$ preserve $S^n_{k,S}(\boldsymbol{\Gamma}, \overline{\mathbb{Q}})$. That is, we have a decomposition

$$S^n_{k,S}(\boldsymbol{\Gamma}, \overline{\mathbb{Q}}) = V(\boldsymbol{f}) \oplus \boldsymbol{U},$$

where $\boldsymbol{U}$ is a $\overline{\mathbb{Q}}$-vector space orthogonal to $V(\boldsymbol{f})$. Therefore, without loss of generality, we may assume that $\boldsymbol{g} \in V(\boldsymbol{f})$.

Now consider a character χ of conductor $\mathfrak{f}_\chi \neq \mathfrak{o}$ such that $\chi_{\boldsymbol{a}}(x) = \mathrm{sgn}_{\boldsymbol{a}}(x)^k$, $\chi^2 \neq 1$ and $G(\chi, \mu - n - l/2) \in \overline{\mathbb{Q}}^\times$, where $G(\chi, \mu - n - l/2)$ is as in equation (8). The existence of such a character follows from the fact that $G(\chi, 2s - n - l/2)$ is the ratio of products of finitely many Euler polynomials.

We recall that if $\tilde{\boldsymbol{f}} \in V(\boldsymbol{f})$, then so is $\tilde{\boldsymbol{f}}^c \in V(\boldsymbol{f})$ and their L-functions agree. In particular, up to some non-zero algebraic number, the identity (7) becomes:

$$\Lambda^{2n}_{k-l/2,\mathfrak{c}}(\mu/2 - l/4, \chi\psi_S) vol(A) < (E|_{k,S}\boldsymbol{\rho})(\mathrm{diag}[z_1, z_2], \mu/2; \chi), (\tilde{\boldsymbol{f}}^c|_{k,S}\boldsymbol{\eta}_n)^c(z_2) > \\ =^{\overline{\mathbb{Q}}^\times} c_{S,k}(\mu/2 - k/2) \boldsymbol{\Lambda}(\mu/2, \boldsymbol{f}, \chi) \tilde{\boldsymbol{f}}^c(z_1).$$

By Theorem 5.5, $\Lambda^{2n}_{k-l/2,\mathfrak{c}}(\mu/2 - l/4, \chi\psi_S) E(z, \mu/2; \chi) \in \pi^\beta N^{2n,r}_{k,S}(\overline{\mathbb{Q}})$ for $\beta \in \mathbb{N}$, and hence the same holds for

$$\Lambda^{2n}_{k-l/2,\mathfrak{c}}(\mu/2 - l/4, \chi\psi_S) E(z, \mu/2; \chi)|_{k,S}\boldsymbol{\rho}.$$

In particular,

$$\pi^{-\beta} \Lambda^{2n}_{k-l/2,\mathfrak{c}}(\mu/2 - l/4, \chi\psi_S)(E|_{k,S}\boldsymbol{\rho})(\mathrm{diag}[z_1, z_2], \mu/2; \chi) = \sum_i \boldsymbol{f}_i(z_1) \boldsymbol{g}_i(z_2),$$

where $\boldsymbol{f}_i, \boldsymbol{g}_i \in N^{n,r}_{k,S}(\overline{\mathbb{Q}})$ by Lemma 5.6. Moreover, $vol(A) = \pi^{d_0} \mathbb{Q}^\times$, where d_0 is the dimension of $\mathbb{H}^d_n$ since the volume of the Heisenberg part is normalized to one. Furthermore,

$$c_{S,k}(\mu/2 - k/2) \in \pi^\delta \overline{\mathbb{Q}}^\times, \;\; \delta \in \frac{1}{2}\mathbb{Z}.$$

Altogether we obtain

$$\sum_i \boldsymbol{f}_i(z_1) < \boldsymbol{g}_i(z_2), \boldsymbol{g}(z_2) > =^{\overline{\mathbb{Q}}^\times} \pi^{\delta - d_0 + \beta} \boldsymbol{\Lambda}(\mu/2, \boldsymbol{f}, \chi) \tilde{\boldsymbol{f}}^c(z_1),$$

where $\boldsymbol{g} := (\tilde{\boldsymbol{f}}^c|_{k,S}\boldsymbol{\eta}_n)^c = \tilde{\boldsymbol{f}}|_{k,S}\boldsymbol{\eta}_n^{-1} \in S^n_{k,S}(\overline{\mathbb{Q}})$. Considering the Fourier expansion of $\boldsymbol{f}_i$'s and $\boldsymbol{f}$, and comparing any (r,t) coefficients for which $c(r,t;\tilde{\boldsymbol{f}}^c) \neq 0$, we find that

$$< \sum_i \alpha_{i,r,t}\boldsymbol{g}_i(z_2), \boldsymbol{g}(z_2) > =^{\overline{\mathbb{Q}}^\times} \pi^{\delta-d_0+\beta}\boldsymbol{\Lambda}(\mu/2,\boldsymbol{f},\chi) \neq 0$$

for some $\alpha_{i,r,t} \in \overline{\mathbb{Q}}$, where the non-vanishing follows from (6), a corollary to Theorem 3.1. Setting $\boldsymbol{h}_{r,t}(z_2) := \sum_i \alpha_{i,r,t}\boldsymbol{g}_i(z_2) \in N^{n,r}_{k,S}(\overline{\mathbb{Q}})$, we obtain

$$< \boldsymbol{h}_{r,t}(z_2), \boldsymbol{g}(z_2) > =^{\overline{\mathbb{Q}}^\times} \pi^{\delta-d_0+\beta}\boldsymbol{\Lambda}(\mu/2,\boldsymbol{f},\chi) \neq 0,$$

or,

$$< \mathfrak{p}^0(\boldsymbol{h}_{r,t}|_{k,S}\boldsymbol{\eta}_n)(z_2), \tilde{\boldsymbol{f}}(z_2) > =^{\overline{\mathbb{Q}}^\times} \pi^{\delta-d_0+\beta}\boldsymbol{\Lambda}(\mu/2,\boldsymbol{f},\chi) \neq 0,$$

where

$$\mathfrak{p}^0 := \begin{cases} \mathfrak{p}, & k \text{ not parallel,} \\ \mathfrak{q}\circ\mathfrak{p}, & k \text{ parallel.} \end{cases}$$

That is, since $\tilde{\boldsymbol{f}} \in V(\boldsymbol{f})$ was arbitrary, the forms $\widetilde{\boldsymbol{h}}_{r,t} := \mathfrak{p}^0(\boldsymbol{h}_{r,t}|_{k,S}\boldsymbol{\eta}_n) \in S^n_{k,S}(\overline{\mathbb{Q}})$ (or rather their projections to $V(\boldsymbol{f})$) for the various (r,t) span the space $V(\boldsymbol{f})$ over $\overline{\mathbb{Q}}$ and

$$< \widetilde{\boldsymbol{h}}_{r,t}, \tilde{\boldsymbol{f}} > \in \pi^{\delta-d_0+\beta}\boldsymbol{\Lambda}(\mu/2,\boldsymbol{f},\chi)\overline{\mathbb{Q}}^\times.$$

That is, for any $\boldsymbol{g} \in V(\boldsymbol{f})$ we have $< \boldsymbol{g}, \boldsymbol{f} > \in \pi^{\delta-d_0+\beta}\boldsymbol{\Lambda}(\mu/2,\boldsymbol{f},\chi)\overline{\mathbb{Q}}^\times$. In particular, the same holds for $\boldsymbol{g} = \boldsymbol{f}$, and that concludes the proof. $\square$

Proof of Theorem 4.6. We follow the same steps as in the proof of Theorem 5.7 but this time we set $s = \sigma/2$. In exactly the same way as above we obtain

$$< \boldsymbol{h}_{r,t}(z_2), \boldsymbol{f}(z_2) > =^{\overline{\mathbb{Q}}^\times} \pi^{\delta-d_0+\beta}\boldsymbol{\Lambda}(\sigma/2,\boldsymbol{f},\chi),$$

for some $\boldsymbol{h}_{r,t} \in \boldsymbol{N}^n_{k,S}(\overline{\mathbb{Q}})$. Thanks to Theorem 5.7 the proof will be finished after dividing the above equality by $< \boldsymbol{f}, \boldsymbol{f} >$ if we make the powers of π precise. Recall that

$$\begin{aligned} c_{S,k}(\sigma/2 - k/2) &=^{\overline{\mathbb{Q}}^\times} \pi^{dn(n+1)/2}\prod_{v\in\mathbf{a}}\frac{\Gamma_n(\sigma/2+k_v-l/2-(n+1)/2)}{\Gamma_n(\sigma/2+k_v-l/2)} \\ &= \pi^{dn(n+1)/2}\prod_{v\in\mathbf{a}}\frac{\prod_{i=0}^{n-1}\Gamma(\sigma/2+k_v-l/2-(n+1)/2-i/2)}{\prod_{i=0}^{n-1}\Gamma(\sigma/2+k_v-l/2-i/2)} =^{\overline{\mathbb{Q}}^\times} \pi^{dn(n+1)/2}. \end{aligned}$$

Hence, $\delta = dn(n+1)/2$. However, this is also equal to the dimension of the space $\mathbb{H}^d_n$, which we denoted by d_0. We are then left only with β, which is provided by Theorem 5.5; namely,

$$\beta = n\sum_{v\in\mathbf{a}}(k_v - l + \sigma) - de,$$

where $e := n^2 + n - \sigma + l/2$ if $2\sigma - l \in 2\mathbb{Z}$ and $\sigma \geq 2n + l/2$, and $e := n^2$ otherwise. This concludes the proof of the theorem. $\square$

References

[1] T. Arakawa, Jacobi Eisenstein Series and a Basis Problem for Jacobi Forms, *Comment. Math. Univ. St. Paul.* 43 (1994), no. 2, 181-216.

[2] Th. Bouganis, J. Marzec, On the standard L-function attached to Siegel-Jacobi modular forms of higher index, available from https://arxiv.org/abs/1706.07287

[3] Th. Bouganis, J. Marzec, On the analytic properties of the standard L-function attached to Siegel-Jacobi modular forms, preprint, available from http://www.maths.dur.ac.uk/users/athanasios.bouganis/Jacobi-analytic.pdf

[4] P. Deligne, Valeurs de fonctions L et périodes d'intégrales. *Proc. Symp. Pure Math.* AMS 33 (1979), 313-346.

[5] M. Eichler, D. Zagier, The theory of Jacobi forms, Progress in Mathematics, volume 55, Springer, 1985.

[6] J. Kramer, An arithmetic theory of Jacobi forms in higher dimensions, *J. reine angew. Math.* 458 (1995), 157-182.

[7] J. Milne, Canonical Models of (Mixed) Shimura Varieties and Automorphic Vector Bundles, *Automorphic forms, Shimura varieties, and L-functions*, Vol. I (Ann Arbor, MI, 1988), 283-414, Perspect. Math., 10, Academic Press, Boston, MA, 1990.

[8] A. Murase, L-functions attached to Jacobi forms of degree n Part I. The basic identity, *J. reine angew. Math.* 401 (1989), 122-156.

[9] A. Murase, L-functions attached to Jacobi forms of degree n Part II. Functional Equation, *Math. Ann.* 290 (1991), 247-276.

[10] A. J. Scholl, Remarks on special values of L-functions, *L-functions and Arithmetic, Proceedings of the Durham Symposium*, July, 1989 (J. Coates and M. J. Taylor, eds.), London Mathematics Society Lecture Notes Series, vol. 153, Cambridge University Press, 1991, 373-392.

[11] G. Shimura, On certain reciprocity-laws for theta functions and modular forms, Acta math. 141 (1978), 35-71. 539-584.

[12] G. Shimura, Euler Products and Eisenstein Series, Conference Board of the Mathematical Sciences (CBMS), Number 93, AMS, 1996.

[13] G. Shimura, Arithmeticity in the Theory of Automorphic Forms, Mathematical Surveys and Monographs, AMS, vol 82, 2000.

[14] N.-P.Skoruppa, D. Zagier, Jacobi forms and a certain space of modular forms, Invent. math. 94, 113-146 (1988)

[15] J. Sturm, The critical values of zeta functions associated to the symplectic group, *Duke Math. J.*, volume 48, 2 (1981), 327-350.

[16] H. Yoshida, Motives and Siegel Modular Forms, *American Journal of Mathematics* Vol. 123, no. 6 (2001), 1171-1197.

[17] C. Ziegler, Jacobi Forms of Higher Degree, *Abh. Math. Sem. Univ. Hamburg* 59 (1989), 191-224.

Department of Mathematical Sciences, Durham University, Durham, UK.
Institute of Mathematics, University of Silesia, Katowice, Poland

E-mail address: athanasios.bouganis@durham.ac.uk, jolanta.marzec@us.edu.pl