On the Centre of Mass of a Random Walk

Chak Hei Lo

Department of Mathematical Sciences, University of Durham, UK.

c.h.lo@durham.ac.uk

Joint work with

Andrew R. Wade

RSC 2017 Durham

Intoduction

Motivation

- Many stochastic processes arising in applications exhibit a range of possible behaviours depending the values of certain key parameters.
- The problem originate from a conjecture by *Paul Erdös* [2].
- The case of *simple symmetric random walk* is solved by Grill [2].
- We extend the result for a much larger class of random walk, with only *minor* moment assumptions.
- An Application in Physics is the *random polymer chain model*. The growth process is repelled or attracted by the centre of mass, depending if it is a poor or good solvent [1].

The centre of mass of a random walk

- Let $d \geq 1$. Let $X, X_1, X_2, \dots$ be a sequence of i.i.d. random variables on $\mathbb{R}^d$.
- Consider the random walk $(S_n, n \in \mathbb{Z}_+)$ in $\mathbb{R}^d$ defined by

$$S_0 := \mathbf{0} \quad \text{and} \quad S_n := \sum_{i=1}^{n} X_i \quad (n \geq 1).$$

- Our object of interest is the *centre of mass process* $(G_n, n \in \mathbb{Z}_+)$ corresponding to the random walk, defined by

$$G_0 := \mathbf{0} \quad \text{and} \quad G_n := \frac{1}{n}\sum_{i=1}^{n} S_i \quad (n \geq 1).$$

Notation and assumptions

Throughout we use the notation

$$\boldsymbol{\mu} := \mathbb{E}\, X, \quad M := \mathbb{E}[(X-\boldsymbol{\mu})(X-\boldsymbol{\mu})^{\top}]$$

whenever the expectations exist; when defined, M is a symmetric d by d matrix. Now we need the following moment assumptions to proceed.

($\boldsymbol{\mu}$) Suppose that $\mathbb{E}\,\|X\| < \infty$.

(M) Suppose that $\mathbb{E}[\|X\|^2] < \infty$ and M is positive-definite.

For our main results, we assume that X has a lattice distribution.

(L) Suppose that X is non-degenerate. Suppose that for a constant vector $\mathbf{b} \in \mathbb{R}^d$ and a d by d matrix H with $|\det H| = h > 0$, we have $\mathbb{P}(X \in \mathbf{b} + H\mathbb{Z}^d) = 1$.

Asymptotic analysis

Strong law of large numbers

Using standard techniques on functional limit theorems, we get the following *strong law of large numbers*.

Proposition 1 (L., Wade, 2017). *If* ($\boldsymbol{\mu}$) *holds, then, as* $n \to \infty$,

$$n^{-1} G_n \to \frac{1}{2}\boldsymbol{\mu}, \text{ a.s.}.$$

Central limit theorem

With the help of Lindeberg–Feller theorem for triangular arrays, we have the following *central limit theorem*.

Proposition 2 (L., Wade, 2017)**.** *If* (M) *holds, then, as* $n \to \infty$,

$$n^{-1/2}\left(G_n - \frac{n}{2}\boldsymbol{\mu}\right) \xrightarrow{d} \mathcal{N}_d(\mathbf{0}, M/3).$$

Local central limit theorem

For $\mathbf{x} \in \mathbb{R}^d$, define $p_n(\mathbf{x}) := \mathbb{P}(n^{-1/2}G_n = \mathbf{x})$, and

$$n(\mathbf{x}) := \frac{\exp\{-\frac{3}{2}\mathbf{x}^{\top}M^{-1}\mathbf{x}\}}{(2\pi)^{d/2}\sqrt{\det(M/3)}}.$$

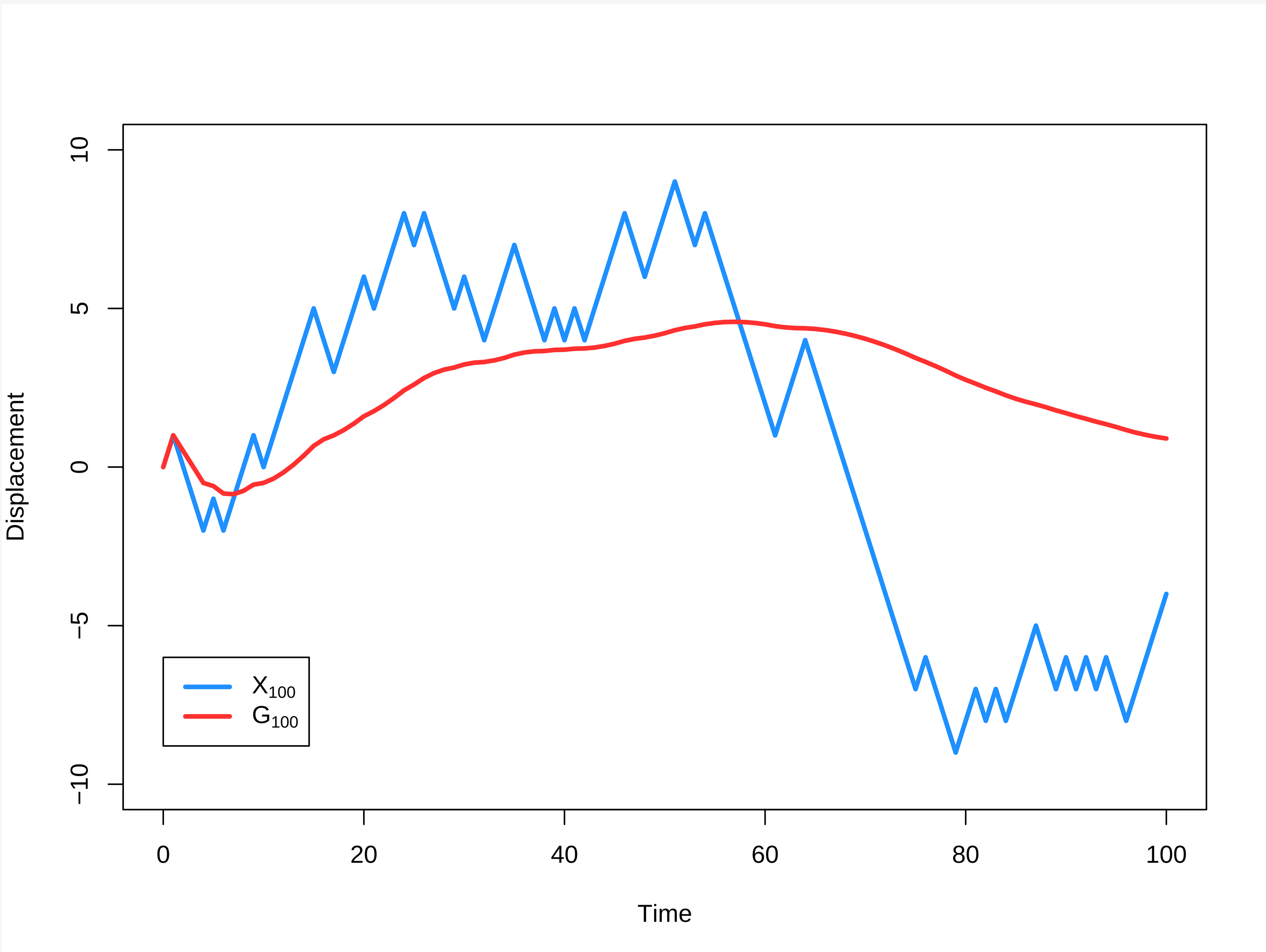
Also define

$$\mathcal{L}_n := \left\{ n^{-3/2}\left(\tfrac{1}{2}n(n+1)\mathbf{b} + H\mathbb{Z}^d\right)\right\}.$$

Our first main result is a *local* central limit theorem.

Theorem 1 (L., Wade, 2017)**.** *Suppose that* (M), (L), *and some technical assumptions hold. Then, as* $n \to \infty$, *we have*

$$\lim_{n\to\infty} \sup_{\mathbf{x}\in\mathcal{L}_n} \left| \frac{n^{3d/2}}{h} p_n(\mathbf{x}) - n\left(\mathbf{x} - \frac{(n+1)}{2n^{1/2}}\boldsymbol{\mu}\right)\right| = 0. \quad (1)$$



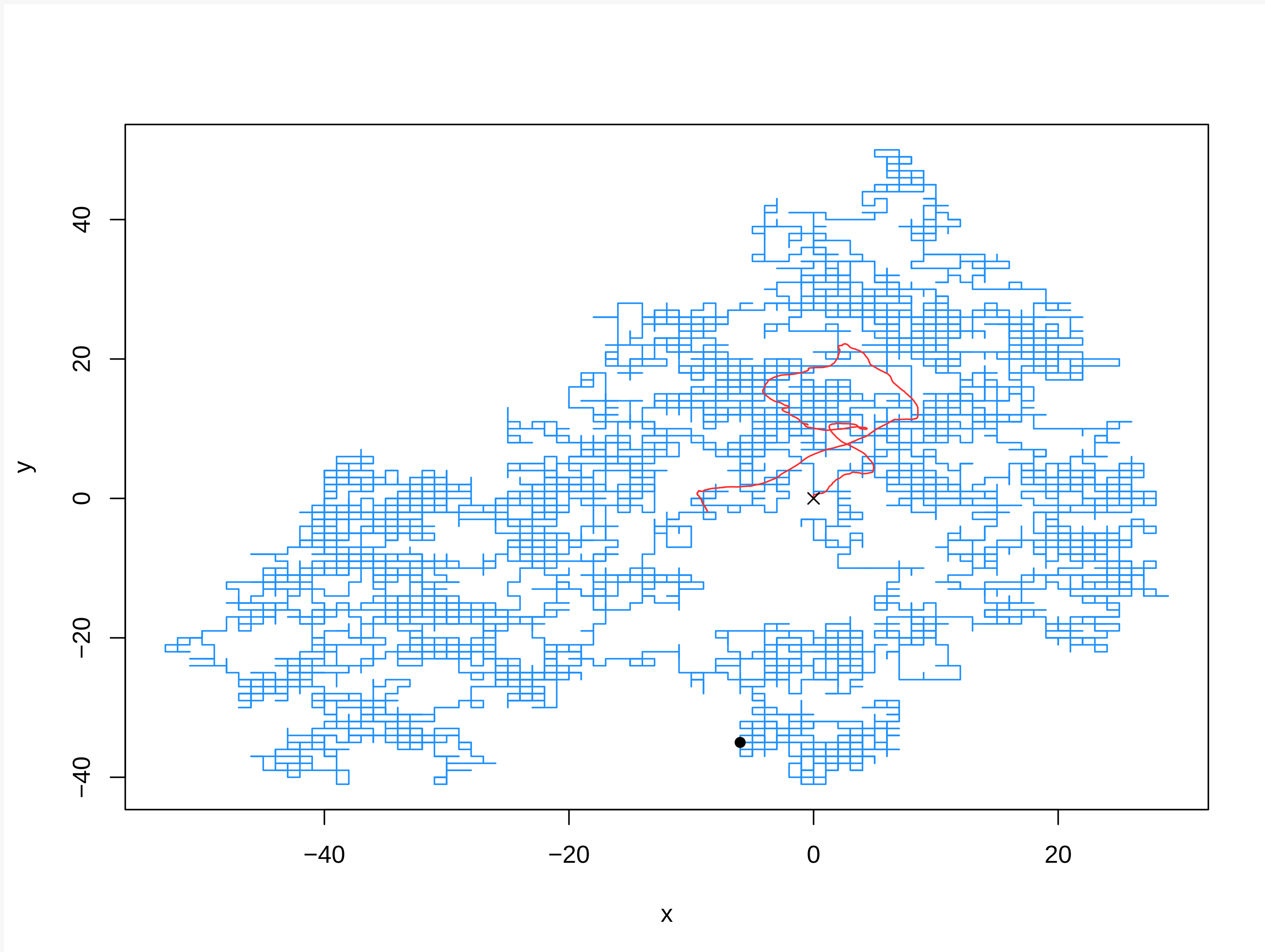


Figure 1: Two simulations of the centre of mass (red) and the corresponding random walk (blue) in one dimension, 100 steps (left) and two dimensions, 10000 steps (right)

Recurrence classification

One dimension

- Depending on different moment assumptions, we can get very different recurrence behaviour of the process. First we give a recurrence result in one dimension.
- In the case of SSRW the fact that G_n returns infinitely often to a neighbourhood of the origin is due to Grill [2, Theorem 1].

Theorem 2 (L., Wade, 2017)**.** *Suppose that* $d = 1$ *and that either of the following two conditions holds.*

(i) Suppose that $\mathbb{E}\,|X| \in (0, \infty)$ *and* $X \overset{d}{=} -X$.

(ii) Suppose that (M) *holds and that* $\mathbb{E}\,X = 0$.

Then we have $\liminf_{n\to\infty} G_n = -\infty$, $\limsup_{n\to\infty} G_n = +\infty$ *and* $\liminf_{n\to\infty} |G_n| = 0$.

On the other hand, if the first moment does not exist, G_n may be transient. The condition we assume is as follows.

(S) Suppose that $X \overset{d}{=} -X$ and X is in the domain of normal attraction of a symmetric α-stable distribution with $\alpha \in (0, 1)$.

Theorem 3 (L., Wade, 2017)**.** *Suppose that* $d = 1$ *and* (L) *holds, i.e.,* $\mathbb{P}(X \in b + h\mathbb{Z}^d) = 1$ *for* $b \in \mathbb{R}$ *and* $h > 0$. *Under some technical conditions we have* $\lim_{n\to\infty} |G_n| = \infty$.

Two dimensions or more

- We have the following transience result in dimensions greater than one.
- We gives a diffusive rate of escape, implying $\lim_{n\to\infty} \|G_n\| = +\infty$.
- In the case of SSRW the result is due to Grill [2, Theorem 1].

Theorem 4 (L., Wade, 2017)**.** *Suppose that* $d \geq 2$ *and that* (M), (L), *and some technical condition hold, and that* $\boldsymbol{\mu} = \mathbf{0}$. *Then*

$$\lim_{n\to\infty} \frac{\log \|G_n\|}{\log n} = \frac{1}{2}, \text{ a.s.}$$

A conjecture

- Obtaining necessary and sufficient conditions for recurrence and transience of G_n is an open problem.
- For $d \geq 2$, we believe that G_n is always 'at least as transient' as the situation in Theorem 4:

Conjecture 1 (L., Wade, 2017)**.** *Suppose that* $\operatorname{supp} X$ *is not conatined in a one-dimensional subspace of* $\mathbb{R}^d$. *Then*

$$\liminf_{n\to\infty} \frac{\log \|G_n\|}{\log n} \geq \frac{1}{2}, \text{ a.s.}$$

Acknowledgements

The authors are grateful to Ostap Hryniv and Mikhail Menshikov for fruitful discussions on the topic of this poster.

References

[1] F. Comets, M.V. Menshikov, S. Volkov, and A.R. Wade, Random walk with barycentric self-interaction, *J. Stat. Phys.* **143** (2011) 855–888.

[2] K. Grill, On the average of a random walk, *Statist. Probab. Lett.* **6** (1988) 357–361.

[3] I.A. Ibragimov and Y.V. Linnik, *Independent and Stationary Sequences of Random Variables*, Wolters-Noordhoff, Groningen, The Netherlands, 1971.