Amie Wilkinson Exercises

Lecture 2

1. Let $(X, \mathcal{B}, \mu)$ be a probability space, and let $\mathcal{A} \subset \mathcal{B}$ be a *finite* sub-sigma algebra of $\mathcal{B}$: i.e., $\#\mathcal{A} < \infty$.

   (a) Let $\mathcal{P}_{\mathcal{A}}$ be the *atoms* of $\mathcal{A}$; that is, the set of nonempty elements of $\mathcal{A}$ that do not contain other nonempty elements. Show that $\mathcal{P}_{\mathcal{A}}$ is a partition of X.

   (b) Let $B \in \mathcal{B}$. Compute the conditional expectation $E(\chi_B \,|\, \mathcal{A})$, where χ_B is the characteristic function of B.

   (c) Let $f \in L^1(X, \mathcal{B}, \mu)$. Compute $E(f \,|\, \mathcal{A})$.

2. Let $\mathcal{P}_1, \mathcal{P}_2, \ldots$ be a sequence of (mod 0) finite partitions of the circle $\mathbb{R}/\mathbb{Z}$ into intervals with the properties:

   - every element if $\mathcal{P}_n$ is a (mod 0) union of elements of $\mathcal{P}_{n+1}$, and
   - the maximum diameter of elements of $\mathcal{P}_n$ tends to 0 as $n \to \infty$.

   Let $B \in \mathcal{B}_{\mathbb{R}/\mathbb{Z}}$ have positive Lebesgue measure: $\mu(B) > 0$. Prove that there exists a sequence of elements $I_n \in \mathcal{P}_n$ such that

   $$\lim_{n\to\infty} \mu(B|I_n) = 1.$$