Amie Wilkinson Exercises

Lecture 4

(1) For this exercise it is instructive to review Jana Rodriguez Hertz's lecture slides and exercises from last week. Let $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be the doubling map ($f(x) = 2x$), and suppose that $g\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is a C^1 map satisfying

$$d_{C^1}(f,g) := \sup_{x\in\mathbb{R}/\mathbb{Z}} \left(|f(x) - g(x)| + |f'(x) - g'(x)|\right) < 1/4.$$

(a) Prove that $\inf_{x\in\mathbb{R}/\mathbb{Z}} g'(x) > 1$.

(b) Show that there is a *lift* $G\colon \mathbb{R} \to \mathbb{R}$ with the following properties:

(i) $G(0) \in [-1/4, 1/4]$

(ii) $G(x+\ell) = G(x) + 2\ell$, for all $x \in \mathbb{R}$ and $\ell \in \mathbb{Z}$

(iii) The following diagram commutes:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\quad G \quad} & \mathbb{R}\,; \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\ \mathbb{R}/\mathbb{Z} & \xrightarrow{\quad g \quad} & \mathbb{R}/\mathbb{Z} \end{array}$$

in other words, for all $x \in \mathbb{R}$:

$$G(x)\,(\mathrm{mod}\ 1) = g(x\,(\mathrm{mod}\ 1)).$$

Show that G is uniquely defined by (i)-(iii).

(c) Prove that g has a fixed point x_0 (i.e. $g(x_0) = x_0$), whose distance to 0 is at most 1/4. (**Hint:** first show this for G).

(d) Show that

$$\int_{\mathbb{R}/\mathbb{Z}} g'(x)\,dx = 2.$$

(e) Prove that g has degree 2; that is, for every $x \in \mathbb{R}/\mathbb{Z}$:

$$\#g^{-1}(x) = 2.$$

(f) * Show that if $h\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is any C^1 map satisfying

$$\inf_{x\in\mathbb{R}/\mathbb{Z}} h'(x) > 1,$$

then

$$m = \int_{\mathbb{R}/\mathbb{Z}} h'(x)\,dx$$

is an integer with $m \geq 2$, the map h has degree m, and h is conjugate to the map $x \mapsto mx$.

(2) For $a \in \mathbb{C}$, $|a| < 1$, define the *Blaschke product* $g_a \colon \mathbb{C} \to \mathbb{C}$ by

$$g_a(z) := \frac{z(z-a)}{(1-\bar{a}z)}$$

(a) Prove that g_a preserves the unit circle; i.e. if $|z| = 1$ then $|g_a(z)| = 1$.
(b) Compute g_a', and prove that g_a preserves the Lebesgue measure μ on $S^1 = \{z : |z| = 1\}$. (here $d\mu = \frac{1}{2\pi}|dz| = \frac{1}{2\pi}d\theta$, if we write $z = e^{i\theta}$).
(c) Prove that g_a has degree 2 for all $|a| < 1$.

(3) (By popular demand) Let (X, d) be a compact metric space, and let $f \colon X \to X$ be an isometry (i.e., $d(fx, fy) = d(x, y)$, for all $x, y \in X$).
(a) Prove that if f is topologically transitive (i.e. the f-orbit of some point is dense), then X is an abelian group and there exists $y \in X$ such that $f(x) = x + y$, for all $x \in X$.
(b) * Suppose that X is a compact Riemannian manifold and d is induced by the Riemann structure. Prove that for every $x \in X$, the closure of the f-orbit of X is a smooth submanifold of X, diffeomorphic to a finite union of tori (where points are 0-dimensional tori). In particular, if f is topologically transitive, then X is a torus. **Hint:** the isometry group of a Riemannian manifold is a Lie group.