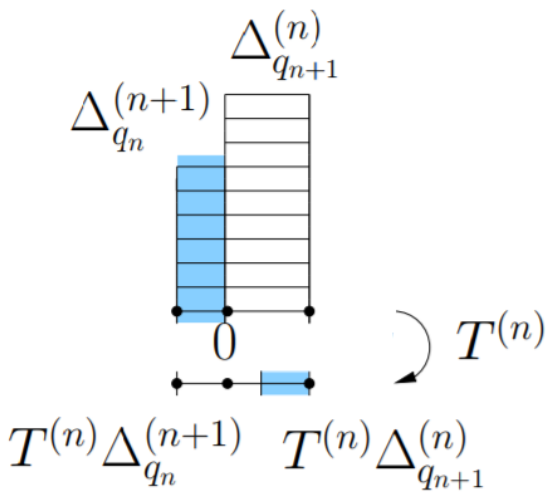
ICTP Summer School on Dynamical Systems
Rotations of the circle and renormalization
Homework 4

Exercise A1 Let R_α be an irrational rotation and consider the inducing intervals $I^{(n)}$ defined in class. Consider the towers representation of R_α as towers with heights q_n and q_{n-1} over the intervals $\Delta^{(n-1)}$ and $\Delta^{(n)}$ (shown in Figure),

(a) Plot inside the towers the orbit segment: $\{R^i_\alpha(0), \quad 0 \le i < q_n + q_{n-1}\}$;

(b) Plot inside the towers the orbit segment: $\{R^i_\alpha(0), \quad 0 \le i < q_n + q_{n+1}\}$;



Exercise A2 Let R_α be a rotation by $\alpha \notin \mathbb{Q}$. Let ξ_n be the partition given by

$$\{\Delta^{(n-1)}_i, 0 \le i < q_n\} \cup \{\Delta^{(n)}_i, 0 \le i < q_{n-1}\}$$

Encode this partition by a word w_n in the letters l (for large) and s (for short) which encode in clockwise order the sequence of short and long intervals (see example in figure).

Show that the strings w_n are obtained recursively from $w_0 = sl$ as follows:

- for n even, get w_{n+1} by replacing each s by l and each l by the word $l^{a_n}s$ (where s^k denote s repeated k times);
- for n odd, get w_{n+1} by replacing each s by l and each l by the word sl^{a_n}.

[A curious fact about this string: it is *almost* palyndrome: if you reverse the order of the letters, it is equal to itself a part possibly the first and the last symbol. Try it out on a few examples! (for the proof see the paper "'A limit theorem for Birkhoff sums..."', Sinai and Ulcigrai, Contemporary Mathematics 2008)]

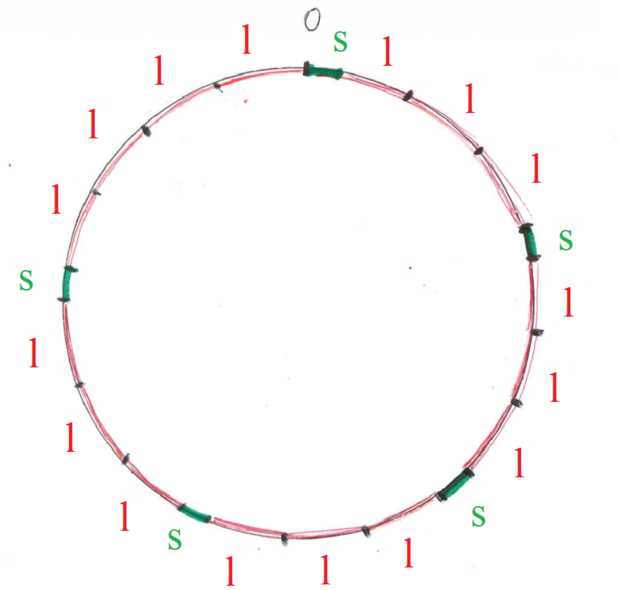


Figure 1: In this example, $w_n = slllslllslllslllsllll$.

Exercise A3 (Kac Lemma)

Let $(X, \mathscr{B}, \mu)$ be a probability space $(\mu(X)1 =)$ and let T be a measure preserving *ergodic* transformation of $(X, \mathscr{B}, \mu)$. Let B be a set such that $\bigcup_{n\geq 0} T^{-n}(B) = X$ (we say that B is a *sweeping* set). Show that

$$\int_Y r_Y d\mu = 1.$$

Hint: You can assume that T is invertible and use (or try to prove) the decomposition of the space as Kakutani towers that we stated in class, namely

$$X = \cup_{n\geq 0} \cup_{k=0}^{n-1} T^k(Y_n), \qquad Y_n := \{y \in Y : \quad r_Y(y) = n\}.$$

Rk: The result is known as Kac's Lemma and shows that the *expected* value of the return time r_Y is $\frac{1}{\mu(Y)}$. It also holds for T non invertible (a proof uses the notion of natural extension).