Dynamical systems

Expanding maps on the circle

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lift

remember

- $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$
- there is a projection $\pi : \mathbb{R} \to \mathbb{S}^1$:

$$x \mapsto [x]$$

lift

lift

- $f : \mathbb{S}^1 \to \mathbb{S}^1$ continuous
- $\Rightarrow \exists\, F : \mathbb{R} \to \mathbb{R}$ continuous
- $$\pi \circ F = f \circ \pi$$
- F unique up to integer traslation
- F is called a lift of f

degree

degree

- F lift of f
- $\Rightarrow F(x+1) - F(x)$ is an integer independient of F, x
- $\deg(f) = F(x+1) - F(x)$ degree of f
- if f homeomorphism, $|\deg(f)| = 1$

degree - proof

proof - degree

- $F(x+1)$ is a lift of f
- since $\pi(F(x+1)) = f(\pi(x+1)) = f(\pi(x))$
- $\Rightarrow F(x+1) - F(x)$ is an integer independent of x

degree - proof

proof - degree

- F, G lifts of f
-

$$\begin{aligned} F(x+1) - F(x) &- (G(x+1) - G(x)) = \\ F(x+1) - G(x+1) &- (F(x) - G(x)) = \\ k &- k = 0 \end{aligned}$$

degree - proof

degree - homeomorphisms

- if $\deg(f) = 0$
- $F(x + 1) = F(x)$ for all $x \in \mathbb{R}$
- $\Rightarrow F$ is not monotone
- $\Rightarrow f$ is not monotone.

degree - proof

degree - homeomorphisms

- if $|\deg(f)| > 1$
- $|F(x+1) - F(x)| > 1$
- $\Rightarrow \exists y \in (x, x+1)$ such that $|F(y) - F(x)| = 1$
- $\Rightarrow f$ is not invertible. $\square$

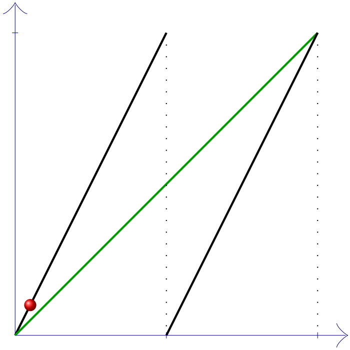
linear expanding maps

a linear expanding map

- $E_2 : \mathbb{S}^1 \to \mathbb{S}^1$ (noninvertible) map
- $$E_2(x) = 2x \pmod 1$$

the map $2x \mod 1$

the map $2x \mod 1$



the map $2x \mod 1$

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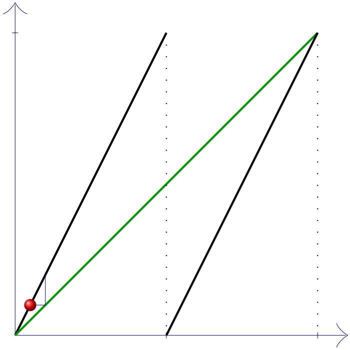

periodic points

number of periodic points

- let us call

$$P_n(f) = \#\{\text{fixed points of } f^n\}$$

number of fixed points

number of fixed points

- $P_n(E_2) = 2^n - 1$
- periodic points of E_2 are dense in $\mathbb{S}^1$

proof

proof

- exercise
- Possible hint. $E_2(z) = z^2$ or $E_2(e^{2\pi i\theta}) = e^{4\pi i\theta}$

other linear expanding maps

other linear expanding maps

- for any integer $m \neq 1$
- $$E_m(x) = mx \pmod 1$$

periodic points

periodic points

- $P_n(E_m) = |m^n - 1|$
- periodic points of E_m are dense in $\mathbb{S}^1$

expanding maps on the circle

expanding maps on the circle

- $f : \mathbb{S}^1 \to \mathbb{S}^1$ is an expanding map on the circle
- if f is continuous and diferentiable
- $$|f'(x)| > 1 \qquad \forall x \in \mathbb{S}^1$$

degree

recall - degree

- the degree of $f : \mathbb{S}^1 \to \mathbb{S}^1$
- is the integer $\deg(f)$ satisfying
- $F(t+1) = \deg(f) + F(t)$
- for any lift $F : \mathbb{R} \to \mathbb{R}$ of f

property

degree and composition

- let $f, g : \mathbb{S}^1 \to \mathbb{S}^1$
- then

$$\deg(g \circ f) = \deg(g)\deg(f)$$

- in particular $\deg(f^n) = \deg(f)^n$

proof

exercise

degree and periodic points

degree and periodic points

- $f : \mathbb{S}^1 \to \mathbb{S}^1$ expanding map
- $\Rightarrow |\deg(f)| > 1$
- and

$$P_n(f) = |\deg(f)^n - 1|$$

proof

proof

- take a lift F of f
- $$|\deg(f)| = |F(x+1) - F(x)| = |F'(\xi)| > 1$$

proof

proof

- it is enough to prove $P_1(f) = |\deg(f) - 1|$:
- $$P_n(f) = P_1(f^n) = |\deg(f^n) - 1| = |\deg(f)^n - 1|$$

proof

proof

- F lift of f
- $\pi(x)$ fixed point of $f \iff F(x) - x \in \mathbb{Z}$
- $G(x) = F(x) - x$ satisfies
- $G(x+1) - G(x) = \deg(f) - 1$
- $\exists$ at least $|\deg(f) - 1|$ points such that $G(\xi) \in \mathbb{Z}$ (the endpoints project into the same)
- $G'(x) \neq 0 \Rightarrow G$ strictly monotone
- $\Rightarrow \exists$ exactly $|\deg(f) - 1|$ fixed points of f in $\mathbb{S}^1$ $\square$

topologically mixing

topologically mixing

- $f : X \to X$ is topologically mixing
- if for any two open sets $U, V \subset X$
- there exists $N > 0$ such that
- $$f^n(U) \cap V \neq \emptyset \qquad \forall n > N$$

rotations

rotations

- rotations are not topologically mixing
- (exercise)

expanding maps

expanding maps

- expanding maps on the circle
- are topologically mixing

proof

proof

- take a lift F of f
- $|F'(x)| \geq \lambda > 1$ for all $x \in \mathbb{R}$
- $|F(b) - F(a)| \geq \lambda |b - a|$
- $|F^n(b) - F^n(a)| \geq \lambda^n |b - a|$
- for all interval I there exists $N > 0$
- such that length$(F^N(I)) > 1$
- $\Rightarrow f^n(\pi(I)) \supset \mathbb{S}^1$ for all $n \geq N$ $\square$