Dynamical systems

Expanding maps on the circle. Semiconjugacy

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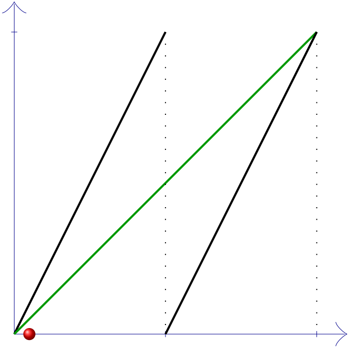
coding

Consider $E_2 : \mathbb{S}^1 \to \mathbb{S}^1$ such that $f(x) = 2x \mod 1$



semiconjugacy

semiconjugacy

- $f : X \to X$ and $g : Y \to Y$ maps
- $h : Y \to X$ is a semiconjugacy from g to f
- if
$$f \circ h = h \circ g$$
- we also say that f is a factor of g

semiconjugacy

semiconjugacy

$$\begin{array}{ccccc} & & g & & \\ & Y & \rightarrow & Y & \\ h & \downarrow & & \downarrow & h \\ & X & \rightarrow & X & \\ & & f & & \end{array}$$

f is a factor of g

E_2 is a factor of σ

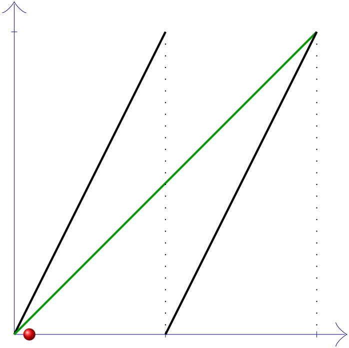
E_2 is a factor of σ

- E_2 is a factor of σ on Σ_2^+
- that is, there exists a continuous surjective h such that
-

$$\begin{array}{ccccc} & & \sigma & & \\ & \Sigma_2^+ & \rightarrow & \Sigma_2^+ & \\ h & \downarrow & & \downarrow & h \\ & \mathbb{S}^1 & \rightarrow & \mathbb{S}^1 & \\ & & E_2 & & \end{array}$$

the semiconjugacy h

Let us define $h : \Sigma_2^+ \to \mathbb{S}^1$



proof

definition of h

- define

$$h(\underline{x}) = \bigcap_{n=0}^{\infty} E_2^{-n}(\Delta_{x_n})$$

proof

h is well defined

- $E_2^{-n}(\Delta_{x_n})$ consists of 2^n intervals of length $\frac{1}{2^{n+1}}$
- $$\bigcap_{n=0}^{N} E_2^{-n}(\Delta_{x_n})$$
  is an interval of length $\frac{1}{2^{N+1}}$
- h is a well-defined function

proof

h is a semiconjugacy

- *h* is continuous (excercise)
- *h* is surjective (excercise)
- $$h \circ \sigma = E_2 \circ h$$

general expanding maps

general expanding maps

- now let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a general expanding map
- suppose $\deg(f) = 2$
- $\Rightarrow$ there is only one fixed point p
- $\Rightarrow$ there is only one point $q \neq p$ such that $f(q) = p$
- call $\Delta_0 = [p, q]$ and $\Delta_1 = [q, p]$

expanding maps are factors of σ

theorem

- $f : \mathbb{S}^1 \to \mathbb{S}^1$ expanding map
- $\deg(f) = 2$
- $\Rightarrow f$ is a factor of σ on Σ_2^+
- $\exists\ h : \Sigma_2^+ \to \mathbb{S}^1$ such that $f^n(h(\underline{x})) \in \Delta_{x_n}$ for all $n \geq 0$

proof

definition of h

- following the previous theorem, let us define

$$h(\underline{x}) = \bigcap_{n=0}^{\infty} f^{-n}(\Delta_{x_n})$$

proof

h is well defined

- $$\bigcap_{n=0}^{N} f^{-n}(\Delta_{x_n}) \neq \emptyset$$
  is an interval (induction)
- $f^n(\xi), f^n(\eta) \in \Delta_{x_n}$ for all n
- $\Rightarrow \xi = \eta$

proof

h is a semiconjugacy

- h is continuous
- h is surjective
- $f \circ h = h \circ \sigma$ $\square$

hints

hints

- define

$$\Delta_{x_0 x_1 \ldots x_N} := \bigcap_{n=0}^{N} f^{-n}(\Delta_{x_n})$$

hints

hints

- prove by induction
- $$\Delta_{x_0 \ldots x_N} = [a_N, b_N]$$
- with $f^{N+1}(a_N) = f^{N+1}(b_N) = p$
- f^{N+1} is injective in (a_N, b_N)