1.1 Measure preserving transformations

In this section we present the definition and many examples of *measure-preserving transformations*. Let $(X, \mathscr{B}, \mu)$ be a measure space. For the ergodic theory part of our course, we will use the notation $T : X \to X$ for the map giving a discrete dynamical system, instead than $f : X \to X$ (T stands for transformation). This is because we will use the letter f for functions $f : X \to \mathbb{R}$ (which will play the role of observables).

Definition 1.1.1. A transformation $T : X \to X$ is *measurable*, if for any measurable set $A \in \mathscr{B}$ the preimage is again measurable, that is $T^{-1}(A) \in \mathscr{B}$.

One can show that if (X, d) is a metric space, $\mathscr{B} = \mathscr{B}(X)$ is the Borel $\sigma-$algebra and $T : X \to X$ is continuous, than in particular T is measurable. All the transformations we will consider will be measurable.

[Even if not explicitly stated, when in the context of ergodic theory we consider a transformation $T : X \to X$ on a measurable space $(X, \mathscr{B})$ we implicitly assume that it is measurable.]

Definition 1.1.2. A transformation $T : X \to X$ is *measure-preserving* if it is measurable and if for all measurable sets

$$\mu(T^{-1}(A)) = \mu(A), \qquad \text{for all } A \in \mathscr{B}. \tag{1.1}$$

We also say that the transformation T *preserves* μ.

If μ satisfies (1.1), we say that the measure μ is *invariant* under the transformation T.

Notice that in (1.1) one uses T^{-1} and not T. This is essential if T is not invertible, as it can be seen in Example 1.1.1 below (on the other hand, one could alternatively use forward images if T is invertible, see Remark 1.1.2 below). Notice also that we need to assume that T is measurable to guarantee that $T^{-1}(A)$ is measurable, so that we can consider $\mu(T^{-1}(A))$ (recall that a measure is defined only on measurable sets).

We will see many examples of measure-preserving transformations both in this lecture and in the next ones.

Remark 1.1.1. Let T be measurable. Let us define $T_*\mu : \mathscr{B} \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$T_*\mu(A) = \mu(T^{-1}(A)), \qquad A \in \mathscr{B}.$$

One can check that $T_*\mu$ is a measure. The measure $T_*\mu$ is called *push-forward* of μ with respect to T. Equivalently, T is measure-preserving if and only if $T_*\mu = \mu$.

Exercise 1.1.1. Verify that if μ is a measure on the measurable space $(X, \mathscr{B})$ and T is a measurable transformation, the push-forward $T_*\mu$ is a measure on $(X, \mathscr{B})$.

Thanks to the extension theorem, to prove that a measure is invariant, it is not necessary to check the measure-preserving relation (1.1) for *all* measurable sets $A \in \mathscr{B}$, but it is enough to check it for a smaller class of subsets:

Lemma 1.1.1. *If the $\sigma-$algebra $\mathscr{B}$ is generated by an algebra $\mathscr{A}$ (that is, $\mathscr{B} = \mathscr{B}(\mathscr{A})$), then μ is preserved by T if and only if*

$$\mu(T^{-1}(A)) = \mu(A), \qquad \text{for all } A \in \mathscr{A}, \tag{1.2}$$

that is, it is enough to check the measure preserving relation for the elements on the generating algebra $\mathscr{A}$ and then it automatically holds for all elements of $\mathscr{B}(\mathscr{A})$.

Proof. Consider the two measures μ and $T_*\mu$. If (1.2) holds, then μ and $T_*\mu$ are equal on the algebra $\mathscr{A}$. Moreover, both of them satisfy the assumptions of the Extension theorem, since they are measures. The uniqueness part of the Extension theorem states that there is a *unique* measure that extends their common values on the algebra. Thus, since μ and $T_*\mu$ are both measures that extend the same values on the algebra, by uniqueness they must coincide. Thus, $\mu = T_*\mu$, which means that T is measure-preserving. The converse is trivial: if μ and $T_*\mu$ are equal on elements of $\mathscr{B}(\mathscr{A})$, in particular they coincide on $\mathscr{A}$. □

As a consequence of this Lemma, to check that a transformation T is measure preserving, it is enough to check it for:

($\mathbb{R}$) *intervals* $[a, b]$ if $X = \mathbb{R}$ or $X = I \subset \mathbb{R}$ is an interval and $\mathscr{B}$ is the the Borel $\sigma-$algebra;

($\mathbb{R}^2$) *rectangles* $[a, b] \times [a, b]$ if $X = \mathbb{R}^2$ or $X = [0, 1]^2$ and $\mathscr{B}$ is the the Borel $\sigma-$algebra;

(S^1) *arcs* if $X = S^1$ with the Borel $\sigma-$algebra;

(Σ) *cylinders* $C_{-m,n}(a_{-m}, \dots, a_n)$ if X is a shift space $X = \Sigma_N$ or $X = \Sigma_A$ and $\mathscr{B}$ is the $\sigma-$algebra;

[This is because *finite unions* of the subsets above mentioned (intervals, rectangles, arcs, cylinders) form algebras of subsets. If one checks that $\mu = T_*\mu$ on these subsets, by additivity of a measure they coincide on the whole algebra of their finite unions. Thus, by the Lemma, μ and $T_*\mu$ coincide on the whole $\sigma-$algebra generated by them, which is in all cases the corresponding Borel $\sigma-$algebra.]

Examples of measure-preserving transformations

Example 1.1.1. [**Doubling map**] Consider $(X, \mathscr{B}, \lambda)$ where $X = [0, 1]$ and λ is the Lebesgue measure on the Borel $\sigma-$algebra $\mathscr{B}$ of X. Let $f(x) = 2x \mod 1$ be the doubling map. Let us check that f preserves λ. Since

$$f^{-1}[a, b] = \left[\frac{a}{2}, \frac{b}{2}\right] \cup \left[\frac{a+1}{2}, \frac{b+1}{2}\right],$$

we have

$$\lambda(f^{-1}[a, b]) = \frac{b-a}{2} + \frac{(b+1)-(a+1)}{2} = b - a = \lambda([a, b]),$$

so the relation (1.1) holds for all intervals. Since if $I = \cup_i I_i$ is a (finite or countable) union of *disjoint* intervals $I_i = [a_i, b_i]$, we have

$$\lambda(I) = \sum_i |b_i - a_i|,$$

one can check that $\lambda(f^{-1}(I)) = \lambda(I)$ holds also for all I which belong to the algebra of finite unions of intervals. Thus, by the extension theorem (see Lemma 1.1.1 and (S^1)), we have $\lambda(f^{-1}(B)) = \lambda(B)$ for all Borel measurable sets.

On the other hand check that $\lambda(f([a, b])) = 2\lambda([a, b])$, so $\lambda(f([a, b])) \neq \lambda([a, b])$. This shows the importance of using T^{-1} and not T in the definition of measure preserving.

Example 1.1.2. [**Rotations**] Let $R_\alpha : S^1 \to S^1$ be a rotation. Let λ be the Lebesgue measure on the circle, which is the same than the $1-$dimensional Lebesgue measure on $[0, 1]$ under the identification of S^1 with $[0, 1]/\sim$. The measure $\lambda(A)$ of an arc is then given by the arc length divided by 2π, so that $\lambda(S^1) = 1$.

Remark that if R_α is the counterclockwise rotation by $2\pi\alpha$, than $R_\alpha^{-1} = R_{-\alpha}$ is the clockwise rotation by $2\pi\alpha$. If A is an arc, it is clear that the image of the arc under the rotation has the same arc length, so

$$\lambda(R_\alpha^{-1}(A)) = \lambda(A), \qquad \text{for all arcs } \; A \subset S^1.$$

Thus, by the Extension theorem (see (S^1) above), we have $(R_\alpha)_*\lambda = \lambda$, that is R_α is measure preserving.

In this Example, one can see that we also have $\lambda(R_\alpha(A)) = \lambda(R_\alpha^{-1}(A)) = \lambda(A)$. This is the case more in general for *invertible* transformations:

Remark 1.1.2. Suppose T is invertible with T^{-1} measurable. Then T preserves μ if and only if

$$\mu(TA) = \mu(A), \qquad \text{for all measurable sets } A \in \mathscr{B}. \tag{1.3}$$

Exercise 1.1.2. Prove the remark, by first showing that if T is invertible (injective and surjective) one has

$$T(T^{-1}(A)) = A, \qquad T^{-1}(T(A)) = A.$$

[Notice that this is false in general if T is not invertible. For any map T one has the inclusions

$$T(T^{-1}(A)) \subset A, \qquad A \subset T^{-1}(T(A)),$$

but you can give examples where the first inclusion can be strict if T is not surjective and the second inclusion $A \subset T^{-1}(T(A))$ is strict if T is not injective.]

In the next example, we will use the following:

Remark 1.1.3. Let $(X, \mathscr{B}, \mu)$ be a measure-space. If $T : X \to X$ and $S : X \to X$ both preserve the measure μ, than also their composition $T \circ S$ preserves the measure μ. Indeed, for each $A \in \mathscr{B}$, since $T^{-1}(A) \in \mathscr{B}$ since T is measurable. Then, using first that S is measure preserving and then that T is also measure preserving, we get

$$\mu(S^{-1}(T^{-1}(A))) = \mu(T^{-1}(A)) = \mu(A).$$

Thus, $T \circ S$ is measure-preserving.

Example 1.1.3. [**Toral automorphisms**] Let $f_A : \mathbb{T}^2 \to \mathbb{T}^2$ be a toral automorphism; A denotes the corresponding invertible integer matrix. Let us show that f_A preserves the 2−dimensional Lebesgue measure λ on the torus. As usual be identify $\mathbb{T}^2$ with the unit square $[0,1)^2$ with oposite sides identified. Since the set of all finite unions of rectangles in $[0,1)^2$ forms an algebra which generates the Borel σ-algebra of the metric space $(\mathbb{T}^2, d)$, and since $f_A^{-1} = f_{A^{-1}}$ is measurable, it is sufficient to prove $\lambda(f_A(R)) = \lambda(R)$ for all rectangles $R \subset [0,1)^2$. The image of R under the linear transformation A is the parallelogram AR. Since $|\det(A)| = 1$, AR has the same area as A. The parallelogram AR can be partitioned into finitely many disjoint polygons P_j, such that for each j we find an integer vector $\mathbf{m_j} \in \mathbb{Z}^2$ with $P_j + \mathbf{m_j} \in [0,1)^2$. Thus

$$f_A(R) = \bigcup_j (P_j + \mathbf{m_j}).$$

Since f_A is invertible, the sets $P_j + \mathbf{m_j}$ are pairwise disjoint, and hence

$$\lambda(f_A(R)) = \sum_j \lambda(P_j + \mathbf{m_j}) = \sum_j \lambda(P_j) = \lambda(R)$$

which completes the proof. (In the second equality above we have used that translations preserve the Lebesgue measure λ.)

Example 1.1.4. [**Gauss map**]

Let $X = [0,1]$ with the Borel $\sigma-$algebra and let $G : X \to X$ be the Gauss map (see Figure 1.1). Recall that $G(0) = 0$ and if $0 < x \leq 1$ we have

$$G(x) = \left\{\frac{1}{x}\right\} = \frac{1}{x} - n \qquad \text{if} \quad x \in P_n = \left(\frac{1}{n+1}, \frac{1}{n}\right].$$

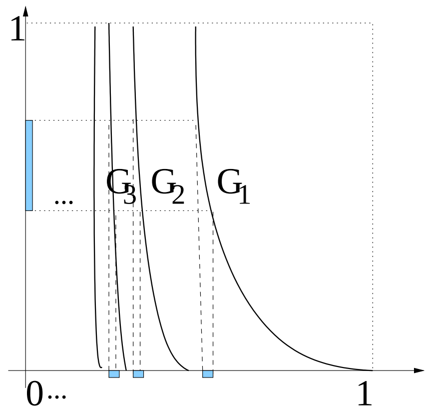

Figure 1.1: The first branches of the graph of the Gauss map.

The *Gauss measure* μ is the measure defined by the density $\frac{1}{(1+x)\log 2}$, that is the measure that assigns to any interval $[a,b] \subset [0,1]$ the value

$$\mu([a,b]) = \frac{1}{\log 2} \int_a^b \frac{1}{1+x} \mathrm{d}x.$$

By the Extension theorem, this defines a measure on all Borel sets. Since

$$\int_0^1 \frac{1}{1+x} = \log(1+x)|_0^1 = \log 2 - \log 1 = \log 2,$$

the factor $\log 2$ in the density is such that $\mu([0,1]) = 1$, so the Gauss measure is a probability measure.

[The Gauss measure was discovered by Gauss who found that the correct density to consider to get invariance was indeed $1/(1+x)$.]

Proposition 1. *The Gauss map G preserves the Gauss measure μ, that is $G_*\mu = \mu$.*

Proof. Consider first an interval $[a,b] \subset [0,1]$. Let us call G_n the branch of G which is given by restricting G to the interval P_n. Since each G_n is surjective and monotone, the preimage $G^{-1}([a,b])$ consists of countably many intervals, each of the form $G_n^{-1}([a,b])$ (see Figure 1.1). Let us compute $G_n^{-1}([a,b])$:

$$\begin{aligned} G_n^{-1}([a,b]) = \{x \text{ s.t. } G_n(x) \in [a,b]\} &= \left\{x \text{ s.t. } a \leq \frac{1}{x} - n \leq b\right\} \\ &= \left\{x \text{ s.t. } \frac{1}{b+n} \leq x \leq \frac{1}{a+n}\right\} = \left[\frac{1}{b+n}, \frac{1}{a+n}\right]. \end{aligned}$$

Remark also that $G_n^{-1}([a,b])$ are clearly all disjoint. Thus, by countably additivity of a measure, we have

$$\begin{aligned}\mu(G^{-1}([a,b])) &= \mu\left(\bigcup_{n=1}^{\infty} G_n^{-1}([a,b])\right) = \mu\left(\bigcup_{n=1}^{\infty}\left[\frac{1}{b+n},\frac{1}{a+n}\right]\right) = \sum_{n=1}^{\infty}\mu\left(\left[\frac{1}{b+n},\frac{1}{a+n}\right]\right)\\ &= \sum_{n=1}^{\infty}\int_{\frac{1}{b+n}}^{\frac{1}{a+n}}\frac{1}{\log 2}\frac{\mathrm{d}x}{(1+x)} = \frac{1}{\log 2}\sum_{n=1}^{\infty}\left(\log\left(1+\frac{1}{a+n}\right)-\log\left(1+\frac{1}{b+n}\right)\right)\\ &= \frac{1}{\log 2}\sum_{n=1}^{\infty}\left(\log\left(\frac{1+a+n}{a+n}\right)-\log\left(\frac{1+b+n}{b+n}\right)\right).\end{aligned}$$

By definition, the sum of the series is the limit of its partial sums and we have that

$$\sum_{n=1}^{N}\log\left(\frac{1+a+n}{a+n}\right) = \sum_{n=1}^{N}\log(1+a+n)-\log(a+n).$$

Remark that the sum is a telescopic sum in which consecutive terms cancel each other (write a few to be convinced), so that

$$\sum_{n=1}^{N}(\log(1+a+n)-\log(a+n)) = -\log(a+1)+\log(1+a+N).$$

Similarly,

$$\sum_{n=1}^{N}(\log(1+b+n)-\log(b+n)) = -\log(b+1)+\log(1+b+N).$$

Thus, going back to the main computation:

$$\begin{aligned}G^{-1}([a,b]) &= \frac{1}{\log 2}\lim_{N\to\infty}\sum_{n=1}^{N}\left(\log\left(\frac{1+a+n}{a+n}\right)-\log\left(\frac{1+b+n}{b+n}\right)\right)\\ &= \frac{1}{\log 2}\lim_{N\to\infty}(\log(1+a+N)-\log(a+1)-(\log(1+b+N)-\log(b+1)))\\ &= \frac{1}{\log 2}\left[\log(b+1)-\log(a+1)+\lim_{N\to\infty}\left(\log\frac{1+a+N}{1+b+N}\right)\right]\\ &= \frac{1}{\log 2}(\log(b+1)-\log(a+1)+0) = \frac{1}{\log 2}\int_a^b\frac{\mathrm{d}x}{\log 2(1+x)}.\end{aligned}$$

This shows that $\mu(A) = G_*\mu(A)$ for all A intervals. By additivity, $\mu(A) = G_*\mu(A)$ on the algebra of finite unions of intervals. Thus, by the Extension theorem (see Lemma 1.1.1), $\mu = G_*\mu$. □

Spaces and transformations in different branches of dynamics

Measure spaces and measure-preserving transformations are the central object of study in ergodic theory. Different branches of dynamical systems study dynamical systems with different properties. In *topological dynamics*, the discrete dynamical systems $f : X \to X$ studied are the ones in which X is a metric space (or more in general, a topological space) and the transformation f is continuous. In *ergodic theory*, the discrete dynamical systems $f : X \to X$ studied are the ones in which X is a *measured space* and the transformation f is *measure-preserving.*

Similarly, other branches of dynamical systems study spaces with different structures and maps which preserves that structure (for example, in *holomorphic dynamics* the space X is a subset of the complex plan $\mathbb{C}$ (or $\mathbb{C}^n$) and the map $f : X \to X$ is a *holomorphic map*; in *differentiable dynamics* the space X is a subset of $\mathbb{R}^n$ (or more in general a *manifold*, for example a surface) and the map $f : X \to X$ is *smooth* (that is differentiable and with continuous derivatives) (as summarized in the Table below) and so on...

branch of dynamics	**space** X	**transformation** $f : X \to X$
Topological dynamics	metric space (or topological space)	continuous map
Ergodic Theory	measure space	measure-preserving map
Holomorphic Dynamics	subset of $\mathbb{C}$ (or $\mathbb{C}^n$)	holomorphic map
Smooth Dynamics	subset of $\mathbb{R}^n$ (or manifold, as surface)	smooth (continuous derivatives)