ICTP Summer School on Dynamical Systems
Rotations of the circle and renormalization
Week 2 - Homework 1

Exercise A1 Let $\mathscr{A} = \{0, 1\}$. Consider the two *Sturmian* substitutions

$$\sigma_0 := \begin{cases} \sigma_0(0) = 0 \\ \sigma_0(1) = 01 \end{cases} \qquad \sigma_1 := \begin{cases} \sigma_1(0) = 10 \\ \sigma_1(1) = 1 \end{cases}$$

(a) Compute the word $u = \sigma_0^2 \sigma_1^4 \sigma_0^3 (11)$ (applying the substitutions to the finite word 11). Consider a biinfinite word w which contains u, i.e. $w = \cdots u \cdots$ and compute its first three derivatives to recover the values $2, 4, 3$ (disregard the endings where you don't have information to derive).

(a') Compute (a part of) $\sigma_0^2(\overline{1})$ (here $\overline{1}$ is the periodic sequence 1), $\sigma_0^2 \sigma_1^4(\overline{0})$ and $\sigma_0^2 \sigma_1^4 \sigma_0^3(\overline{1})$ and verify that they have common central blocks.

(b) From Theorem 1 and Theorem 2 proved in class show that for every square cutting sequence w there exists $(a_n)_{n \in \mathbb{N}}$, $a_n \in \mathbb{N}$ such that

$$w \in \bigcap_{n \in \mathbb{N}} \sigma_0^{a_0} \sigma_1^{a_1} \sigma_0^{a_2} \cdots \sigma_{\epsilon_n}^{a_n} \{0, 1\}^{\mathbb{Z}},$$

where $\epsilon_n = 0$ if n is even, 1 otherwise.

(b') Show also that

$$w = \lim_{n \to \infty} \sigma_0^{a_0} \sigma_1^{a_1} \sigma_0^{a_2} \cdots \sigma_0^{a_{2n}} (\overline{1})$$

where $\overline{1}$ is the periodic sequence 1 and the limit is in the topology on shift spaces, i.e. it means that the sequences share longer and longer central blocks.

[This type of limit is know as an $\mathscr{S}$*-adic* expansion. More in general an $\mathscr{S}$-adic system, where $\mathscr{S}$ is a finite collection of substitutions (here $\mathscr{S} = \{\sigma_0, \sigma_1\}$, consists of all sequences which admit an $\mathscr{S}$-adic expansion (with the shift dynamics).]

Exercise A2

(a) Show that every finite sequence u which is admissible with all its derivatives (when defined) can be realized as a square cutting sequence of a *finite segment* of a line. Deduce that every sequence which is infinitely derivable is in the closure of square cutting sequences.

(b) Find an example(s) of a sequence in $\{0, 1\}^{\mathbb{Z}}$ which is infinitely derivable, but it is not a cutting sequence.