ICTP Summer School on Dynamical Systems
Renormalization in entropy zero dynamics
Week 2 - Homework 4

Exercise C1 Let T be a d-IET. Let α_t and α_b as in class. Assume that $\lambda_{\alpha_t} \neq \lambda_{\alpha_b}$.

(a) Draw an example of an IET where $\lambda_{\alpha_t} > \lambda_{\alpha_b}$ and compute the IET obtained by one step of Rauzy induction (case *top*); do it also for the case $\lambda_{\alpha_b} > \lambda_{\alpha_t}$ (case *bottom*).

(b) Show that for any subinterval $J = [a, b) \subset [0, 1)$ the induced map T_J is an IET of at most $d + 2$ intervals. When it is an IET of d or $d + 1$ intervals?

(c) Let $T^{(n)} : I^{(n)} \to I^{(n)}$ be the induced IETs produced by Rauzy-Veech induction and let $\pi^{(n)}$ be their permutations. Show that the lenghts vector $\lambda^{(n)} = (\lambda_\alpha)_{\alpha\in\mathscr{A}}$ of $T^{(n)}$ and the *return times vector* $h^{(n)} = (h^{(n)}_\alpha)_{\alpha\in\mathbb{A}}$, where

$$h^{(n)}_\alpha := \min\{k \geq 1 : \quad T^k(I^{(n)}_\alpha) \subset I^{(n)}\}.$$

satisfy:

$$\lambda^{(0)} = A_0 A_1 \dots A_n \lambda^{(n+1)}, \qquad h^{(n+1)} = A^t_n \cdots A^t_1 A^t_0 h^{(0)}$$

(for convention $h^{(0)}_\alpha = 1$ for all $\alpha \in \mathscr{A}$), where A^t denote the transpose of the matrix A and the matrices A_n are given by

$$A_n = \begin{cases} I + E_{\alpha_t\alpha_b} & \text{in case top;} \\ I + E_{\alpha_b\alpha_t} & \text{in case bottom;} \end{cases}$$

where I the $d \times d$ identity matrix, $E_{\alpha\beta}$ the matrix with a 1 entry in row α, column β and all other entries equal to 0 and α_t and α_b are the letters of the last two intervals of $T^{(n)}$, i.e. $I^{(n)}_{\alpha_t}$ is the last interval *before* the exchange (t for *top* row in the pictures), while $T(I^{(n)}_{\alpha_b})$ is last *after* applying $T^{(n)}$ (b for *bottom* row in the pictures).

Exercise C2 Let T be a d-IET.

(a) Draw an example (for $d > 2$!) and draw T^2 and T^3.

(b) Show that in general T is an IET of at most $n(d - 1) + 1$ intervals.

(c) Deduce that if $(\alpha_i)_{i\in\mathbb{N}}$ is the itinerary of an orbit $\mathscr{O}^+_T(x)$ of a point $x \in [0, 1)$ w.r.t. the natural coding (i.e. $T^i(x) \in I_{\alpha_i}$ for any $i \in \mathbb{N}$, the *complexity* P of the sequence $(\alpha_i)_{i\in\mathbb{N}}$ satisfies $P(n) \leq n(d - 1) + 1$.

(d) Show that if $(\lambda_\alpha)_{\alpha\in\mathscr{A}}$ are rationally independent, the orbits of the discontinuities of T are all distinct, i.e. there is no $\alpha, \beta \in \mathscr{A}$ and $n \geq 1$ such that $T^n(u_\alpha) = u_\beta$.

(e) Find an example of an IET which is minimal but has connections (i.e. there are $\alpha, \beta \in \mathscr{A}$ and $n \geq 1$ such that $T^n(u_\alpha) = u_\beta$).

[A triple (α, β, n) as in (d) is known as *connection.* An IET with no connections is said to satisfy *Keane's condition* and it is minimal (i.e. Keane's theorem holds for IETs with no connections, which are more general than IETs with rationally independent lenghts, as shown by (d).]