ICTP Summer School on Dynamical Systems
Renormalization in entropy zero dynamics (week 2)
Homework 5

Exercise C1 Let $d = 2$ and let T be a 2-IET with initial lengths $(\alpha, 1-\alpha)$ where $\alpha \in (0,1)$ irrational.

(a) Apply Rauzy-Veech induction to T and verify that if we set

$$k := \begin{cases} \left[\frac{1-\alpha}{\alpha}\right] & \text{if } \alpha < 1/2 \\ \left[\frac{\alpha}{1-\alpha}\right] & \text{if } \alpha < 1/2 \end{cases}.$$

(where $[x]$ denotes the integer part of x), Rauzy-Veech induction starts with either k type top (if $\alpha < 1/2$) or k type top (if $\alpha > 1/2$) moves.

(b) For any $n \in \mathbb{N}$, if $\lambda_A^{(n)}$ and $\lambda_B^{(n)}$ are the lengths of the intervals exchanged by $T^{(n)}$ (n^{th} induced map in the induction), set $\alpha_n \in [0,1]$ to be given by

$$\alpha_n := \begin{cases} \frac{\lambda_A^{(n)}}{\lambda_B^{(n)}} & \text{if } \lambda_A^{(n)} < \lambda_B^{(n)} \\ \frac{\lambda_B^{(n)}}{\lambda_A^{(n)}} & \text{if } \lambda_B^{(n)} < \lambda_A^{(n)} \end{cases}$$

. Show that $\alpha_n = \mathscr{F}^n(\alpha)$ where $\mathscr{F}(\alpha)$ is the Farey map given by

$$F(x) = \begin{cases} \frac{1-x}{x} & \text{if } x > \frac{1}{2} \\ \frac{x}{1-x} & \text{if } x \le \frac{1}{2}. \end{cases}$$

[Recalling Exercise 3.3 of Week 1, this also shows that the acceleration of Rauzy-Veech induction obtained by doing in one go all bottom or all top steps, plus the next one of a different type, corresponds to the Gauss map in the special case $d = 2$. This acceleration is known as *Zorich induction*. One can show that Zorich induction preserves a *finite* invariant measure on the space of $d-$IETs, while Rauzy-Veech induction preserves a *infinite* invariant measure (as the Gauss map preserves the Gauss measure, which is finite, while the Farey map $\mathscr{F}$ preserves a measure with density $1/x$ which is infinite.]

Exercise C2 (Renormalization for Birkhoff sums) Let T be a d-IET (with π irreducible and $(\lambda_\alpha)_{\alpha\in\mathscr{A}}$ rationally independent and let $f : I \to \mathbb{R}$. Let $I^{(n)}$ be the inducing intervals for Rauzy Veech induction and define the sequence of induced functions by

$$f^{(n)}(x) := \sum_{i=0}^{h_\alpha^{(n)}-1} f(T^i(x)) \qquad if\ x \in I_\alpha^{(n)}.$$

[This function associate to a point in the base $I^{(n)}$ the Birkhoff sum of the function f *along the tower*, i.e. along the piece of the orbit that goes from the base point up to the top of the tower to which it belongs. They are also called *special Birkhoff sums.*]

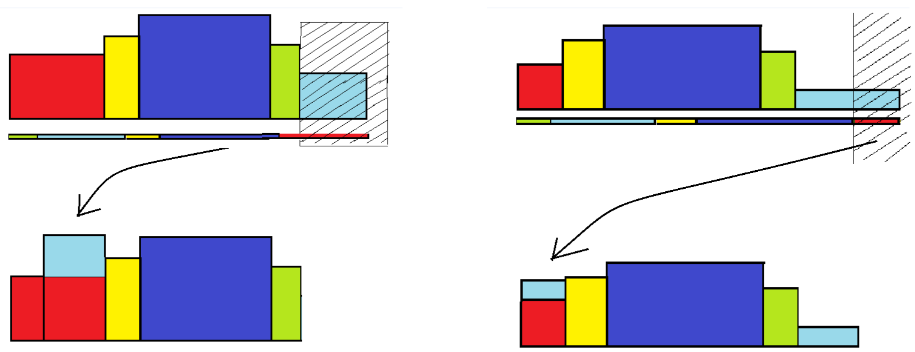
(a) Show that if f is constant on $I_\alpha^{(0)}$ for each $\alpha \in \mathcal{A}$, then for every $n \in \mathbb{N}$ also $f^{(n)}$ is constant on $I_\alpha^{(n)}$ for each $\alpha \in \mathcal{A}$.

(b) Let $f_\alpha^{(n)}$ be the value of $f^{(n)}$ on any point $x \in I_\alpha^{(n)}$ (so that by (a) $f^{(n)} \equiv f_\alpha^{(n)}$ on $I_\alpha^{(n)}$) and let $f^{(n)}$ be the vector with coordinates $f_\alpha^{(n)}$, $\alpha \in \mathcal{A}$. Show that

$$f^{(n+1)} = A_n^t f^{(n)},$$

where A_n is the matrix that transforms lengths vectors in Rauzy-Veech induction (i.e. $f^{(n)}$ transforms as the tower heights/return times vector $h^{(n)}$ (compare with Homework 4 of this week).

[*Hint*: you might want to use cutting and stacking of towers, see e.g. the figure.]



(c) Deduce that the entry $(A_n^t \cdots A_0^t)_{\alpha\beta}$ of the matrix product $A_n^t \cdots A_0^t$ gives the number of time the orbit under T of the interval $I_\beta^{(n)}$ visits the interval $I_\beta^{(0)}$ until the first return time $h_\beta^{(n)}$ to $I^{(n)}$, or equivalently the number of floors of the tower $H_\beta^{(n)}$ which are contained in $I_\alpha^{(0)}$.

(d) Assume that T is periodic under renormalization, i.e. $T^{(n)}$ is equal to T up to rescaling the interval $I^{(n)}$ to unit length (T in this case is self-similar). Show that we can find subspaces

$$F_- \subset F_0 \subset \mathbb{R}^d$$

such that, for every $(f_\alpha)_{\alpha\in\mathcal{A}} \in F_-$, $f^{(n)}$ are exponentially small in n, for every $(f_\alpha)_{\alpha\in\mathcal{A}} \in F_0$ grow at most polynomially, while for $(f_\alpha)_{\alpha\in\mathcal{A}} \in \mathbb{R}^n \backslash F_0$, $f^{(n)}$ grows exponentially.

(e)* For T, f, F_0, F_- as in (d), deduce that if f correspondes to $(f_\alpha)_{\alpha\in\mathcal{A}}$, then

- $(f_\alpha)_{\alpha\in\mathcal{A}} \in F_-$, the Birkhoff sums $S_n f$ are uniformely bounded (i.e. there exists $C > 0$ such that $\|S_n f\|_\infty < C$ for every $n \in \mathbb{N}$;
- if $(f_\alpha)_{\alpha\in\mathcal{A}} \in F_0$, $S_n f$ grow subpolynomially, i.e. for every $\epsilon > 0$ there exists $C > 0$ such that $\|S_n f\|_\infty < C n^\epsilon$ for every $n \in \mathbb{N}$;
- if $(f_\alpha)_{\alpha\in\mathcal{A}} \in \mathbb{R}^d \backslash F_0$, $S_n f$ grow at most polynomially, i.e. there exists $C > 0$ and $\gamma \leq 1$ such that $\|S_n f\|_\infty < C n^\gamma$ for every $n \in \mathbb{N}$.

Hint: You want to decompose $S_n f$ into (a logarithmic number of) special Birkhoff sums.

[Part (d) can be used to show that, among piecewise constant functions, coboundaries (i.e. functions that can be written as $f = g \circ T - g$ for some $g : I \to \mathbb{R}$) are functions f represented by $(f_\alpha)_{\alpha\in\mathcal{A}} \in F^-$ (we say that there are finitely many *obstructions* to solve the cohomological equations). One can also show that while Birkhoff sums of functions with non-zero mean grow linearly, mean zero functions in $\mathbb{R}^d \backslash F_0$ display polynomial deviations of ergodic averages, i.e. there exists $0 < \gamma < 1$ and $c > 0$ such that, for every $x \in [0, 1)$, $S_n f(x) \geq c n^\gamma$ infinitely often.]