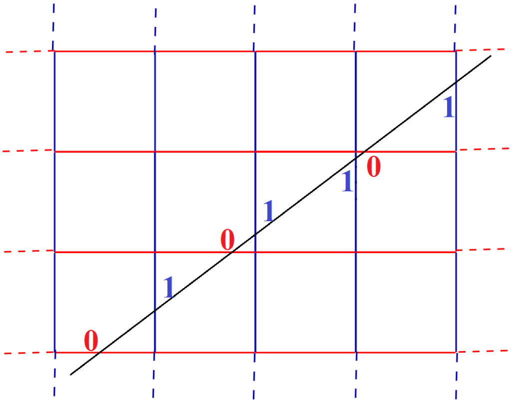
Sturmian sequences as square cutting sequeces



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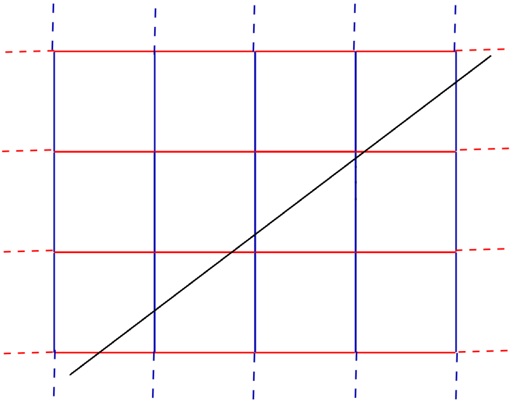

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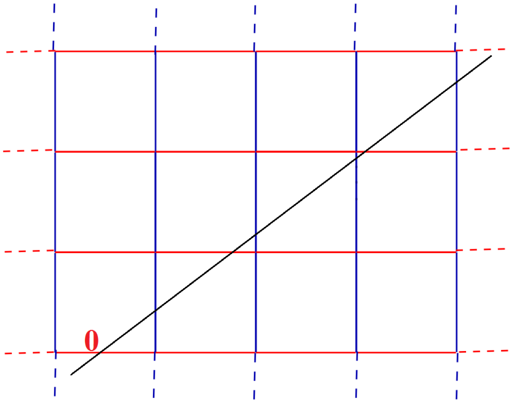

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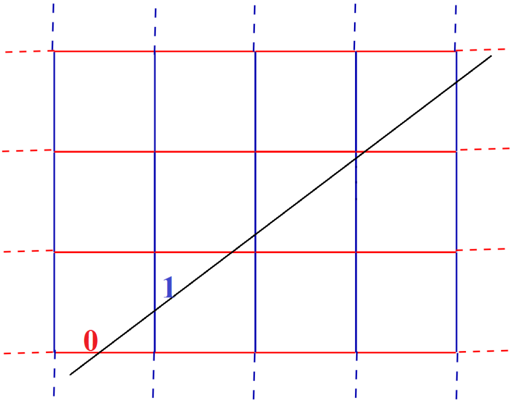

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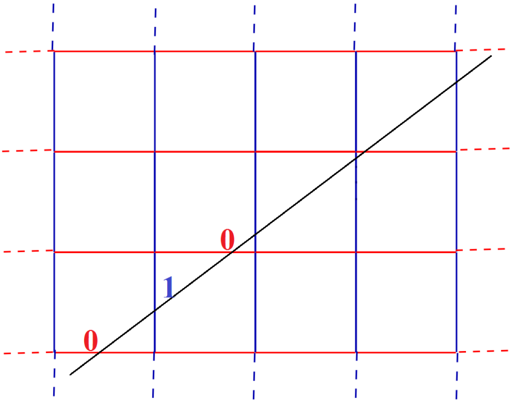

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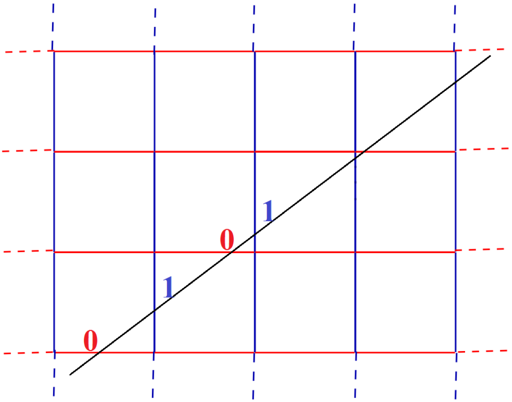

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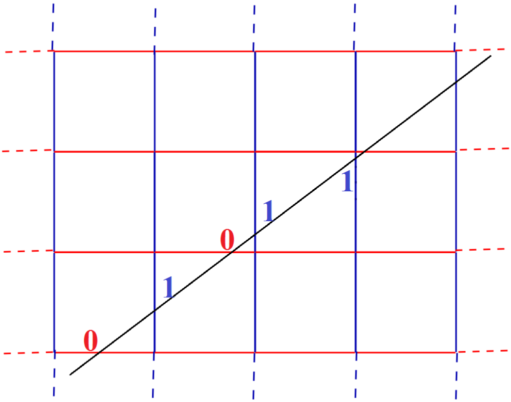

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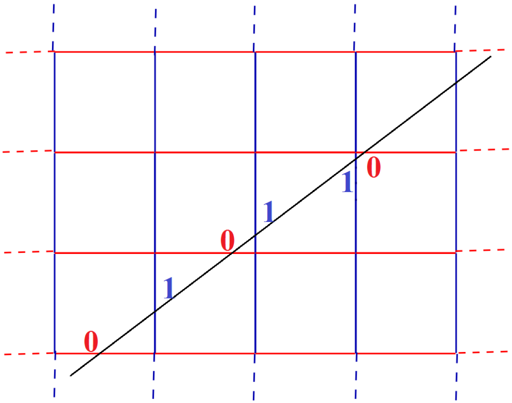

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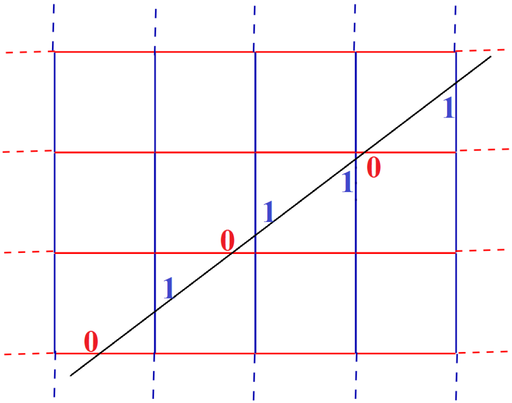

Equivalently: symbolic coding of a linear flow in a square.

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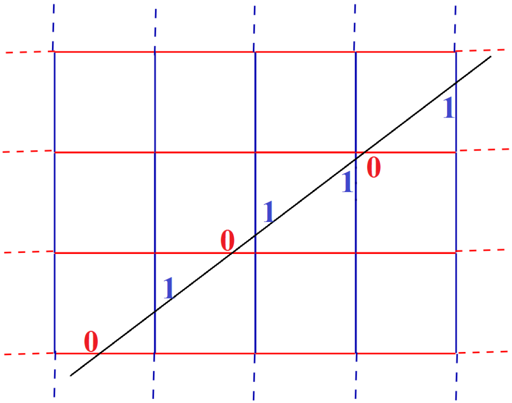

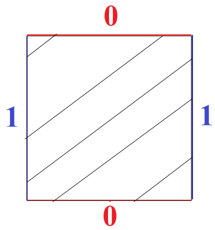

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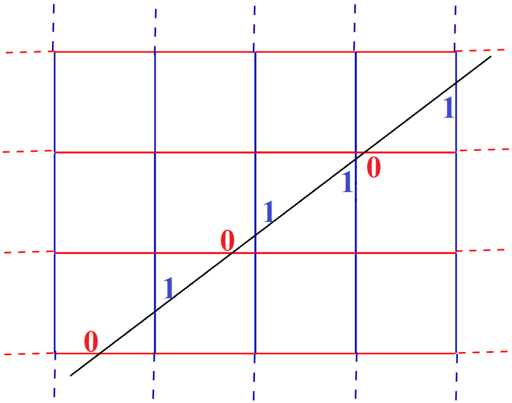

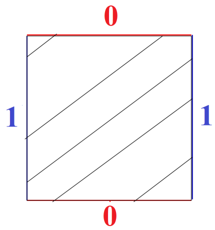

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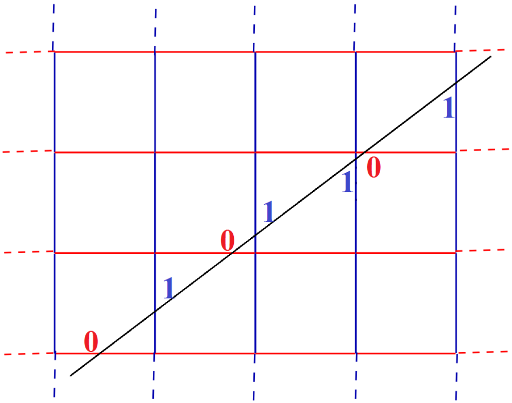

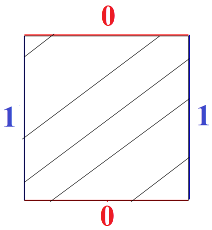

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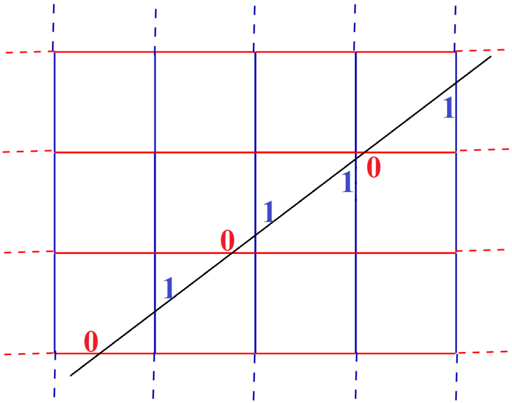

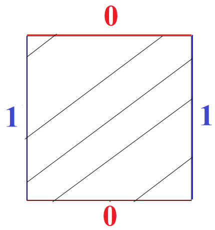

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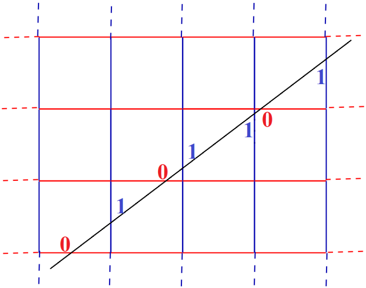
Sturmian sequences complexity

Sturmian sequences are characterized by having the smallest possible *complexity* among non-periodic sequences:

- Let $P(n)$ denote the *number of words of lenght n* which appear in the word w.
- $P(n) = n$ for all n large iff w is periodic (Exercise).
- A sequence is Sturmian iff $P(n) = n + 1$ for all $n \in \mathbb{N}$.



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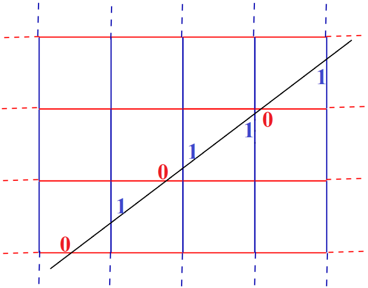

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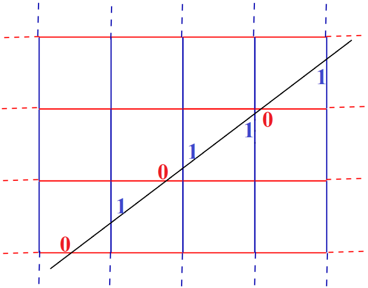

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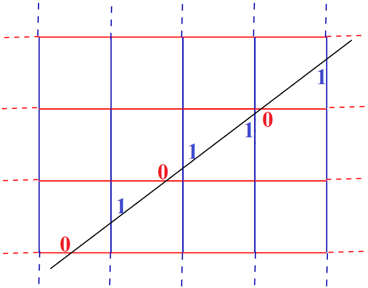

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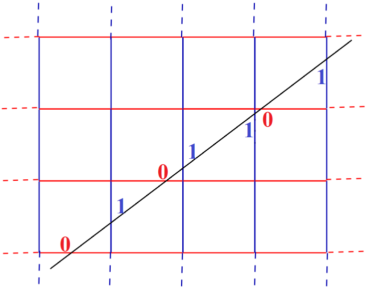

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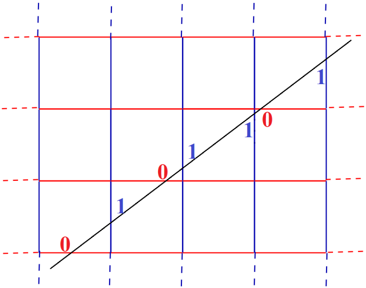

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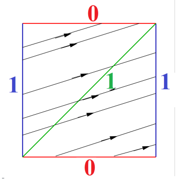
One step of the proof of the key Lemma.

Let w be the cutting sequence in direction $0 \leq \theta < \pi/4$ (*type* 1), e.g.:



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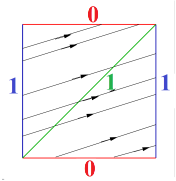

$w = \ldots$ 0 1 1 0 1 1 1 0 1 1 0 1 1 0 1 1 1 $\ldots$

Let us add the diagonal 1.

Let $\tilde{w}$ be the extended sequence:

Each 11 becomes 111; 01 stays 01

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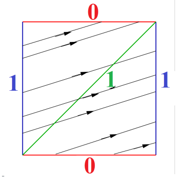

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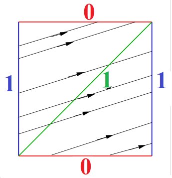

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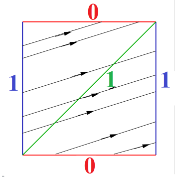

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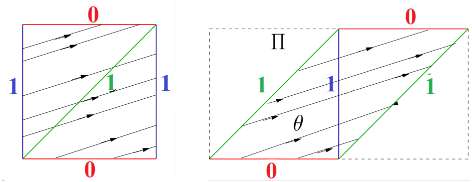

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Let $\tilde{w}$ be the extended sequence:
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Let us cut and paste the rectangle.

Consider the cutting sequence u with respect to the parallelogram Π.
To obtain u from $\tilde{w}$ it is enough to drop the 1s.

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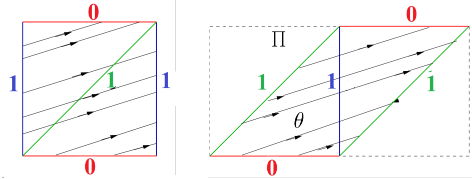

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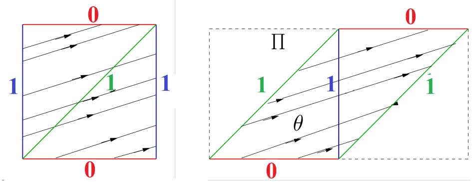

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$$u \;=\; \ldots\; 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ \ldots$$

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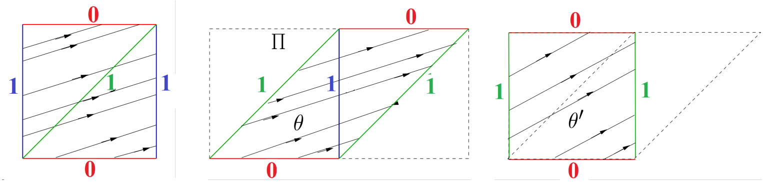

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u = . . . 0 1 0 1 1 0 1 0 1 0 1 1 . . .

Let us add the diagonal 1.

Let $\tilde{w}$ be the extended sequence:
Each 11 becomes 111; 01 stays 01
Let us *renormalize*: we can transform Π in a square by the shear $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Let us transform back the 1s into 1s.

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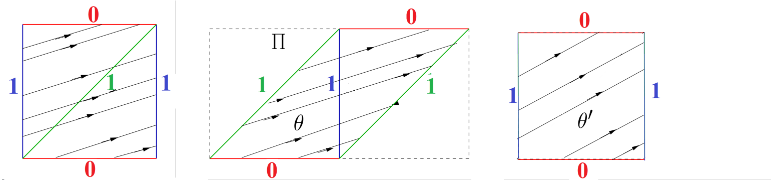

$w = \dots\ 0\ 1\ 1\quad 0\ 1\ 1\ 1\quad 0\ 1\ 1\quad 0\ 1\ 1\quad 0\ 1\ 1\ 1\quad \dots$

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$u = \dots\ 0\quad 1\quad 0\quad 1\quad 1\quad 0\quad 1\quad 0\quad 1\quad 0\quad 1\quad 1\quad \dots$

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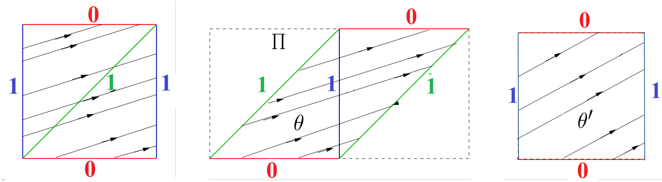

w = ... 0 1 1   0 1 1 1   0 1 1   0 1 1   0 1 1 1 ...

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u = ... 0 1 0 1 1 0 1 0 1 0 1 1 ...

w' = ... 0 1 0 1 1 0 1 0 1 0 1 1 ...

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Remark: The blocks of 1s are now shorter by one.

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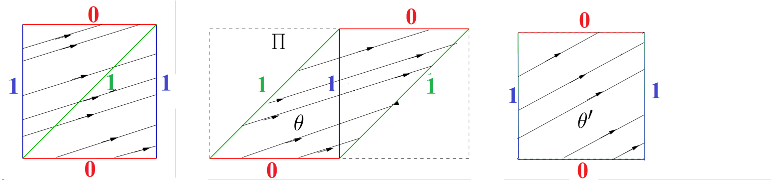

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$\tilde{w}$ = . . . 0 1 1 1 0 1 1 1 1 1 0 1 1 1 0 1 1 1 0 1 1 1 1 1 . . .

u = . . . 0 1 0 1 1 0 1 0 1 0 1 1 . . .

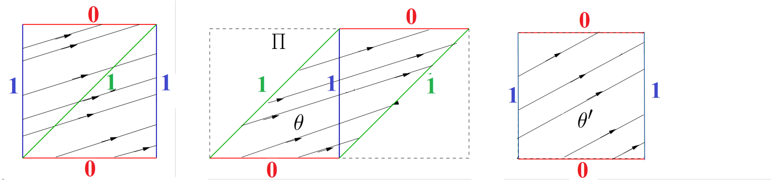
w' = . . . 0 1 0 1 1 0 1 0 1 0 1 1 . . .

Let us add the diagonal 1.

Let $\tilde{w}$ be the extended sequence:
Each 11 becomes 111; 01 stays 01
Let us *renormalize*: we can transform Π in a square by the shear $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Let us transform back the 1s into 1s.
Remark: The blocks of 1s are now shorter by one. Repeat k times to check that the sequence thus obtained is the *derived sequence*.

One step of the proof of the key Lemma.

Let w be the cutting sequence in direction $0 \leq \theta < \pi/4$ (*type* 1), e.g.:



$w = \dots\ 0\ 1\ 1 \quad 0\ 1\ 1\ 1 \quad 0\ 1\ 1 \quad 0\ 1\ 1 \quad 0\ 1\ 1\ 1 \quad \dots$

$\tilde{w} = \dots\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ \dots$

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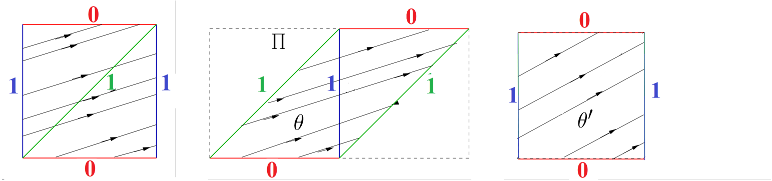

$w \; = \ldots$ 0 1 1   0 1 1 1   0 1 1   0 1 1   0 1 1 1   …

$\tilde{w} \; = \ldots$ 0 1 1 1 0 1 1 1 1 1 0 1 1 1 0 1 1 1 0 1 1 1 1 1 …

$u \; = \ldots$ 0 1 0 1 1 0 1 0 1 0 1 1 …

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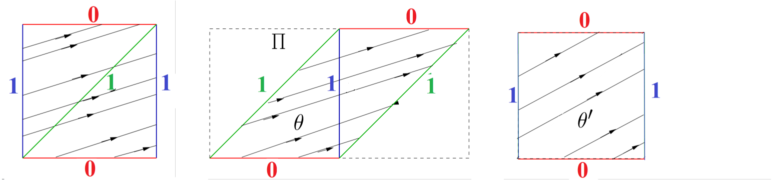

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