A characterization of Sturmian sequences

Corinna Ulcigrai

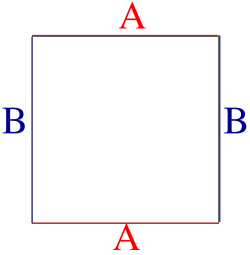
Rk: a slightly different approach than in class (this uses Farey map instead than Gauss map and defines admissibility without value)

ICTP, Trieste, 23 July 2018

The square: isometries and sectors

Let D_4 be the group of isometries of the square.

The letters $\{A, B\}$ are invariant under vertical symmetry and horizontal symmetry and are exachanged if we reflect diagonally



WLOG we can assume that $\theta \in [0, \frac{\pi}{2}]$ and, up to permuting $\{A, B\}$, let $\theta \in \Sigma_0 := [0, \frac{\pi}{4}]$.

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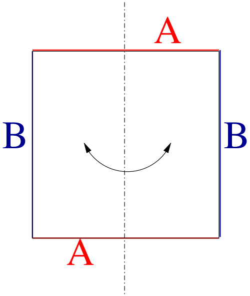

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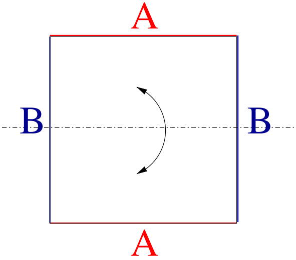

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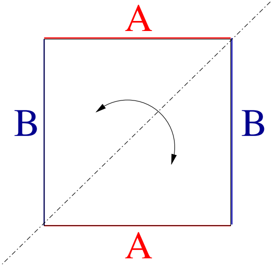

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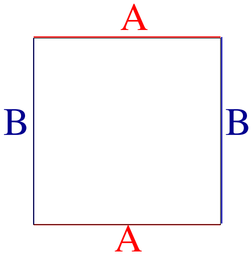

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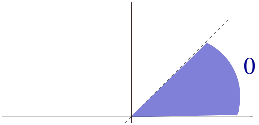

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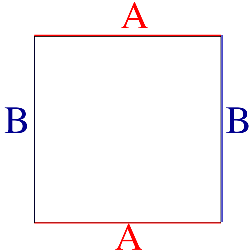

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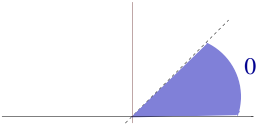

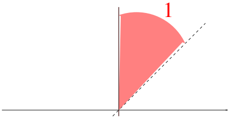

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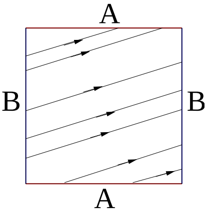
The square: possible transitions

- If $\theta \in \Sigma_0 := \left[0, \frac{\pi}{4}\right]$, AA does not appear:



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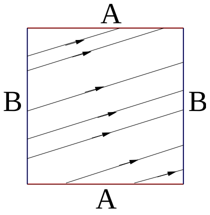

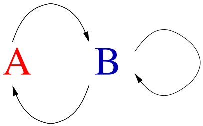

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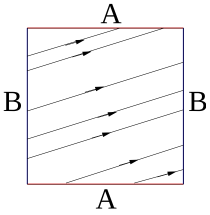

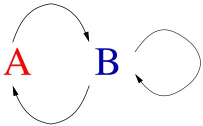

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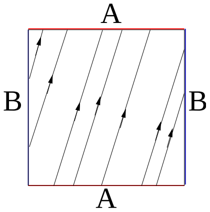
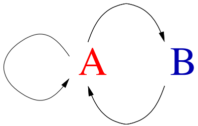
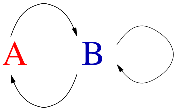




Admissible sequences

Definition

A sequence $w \in \{A, B\}^{\mathbb{Z}}$ is *admissibile* if it gives a infinite path on one of these two diagrams:



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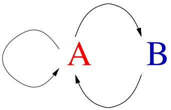

$\mathscr{D}_1$

In this case, we say that w is admissible in $\mathscr{D}_0$ or $\mathscr{D}_1$ respectively.

Lemma

A square cutting sequence is admissible.

If $\theta \in \Sigma_0 = [0, \frac{\pi}{4}]$, w is admissible in $\mathscr{D}_0$, if $\theta \in \Sigma_1 = [\frac{\pi}{4}, \frac{\pi}{2}]$, w is admissibile in $\mathscr{D}_1$.

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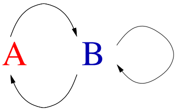

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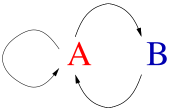

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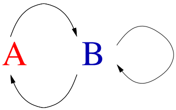

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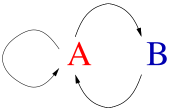

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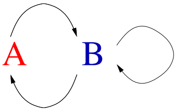

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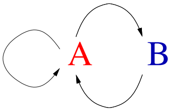

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Derivable sequences

Definition (Derived sequence)

Let w be the cuttings sequence of a trajectory with $\theta \in \Sigma_0$. The derived sequence w' is obtained erasing one B from each block of Bs. If $\theta \in \Sigma_1$, w' is obtained erasing one A from each block of As.

Example

$w = \dots ABBBBABBBABBBBABBBABBB \dots,$
$w' = \dots ABBB \quad ABB \quad ABBB \quad ABB \quad ABB \quad \dots$

Definition (Derivable Sequences)

A sequence $w \in \{A, B\}^{\mathbb{Z}}$ is *derivable* if it is admissible and the derived sequence is admissible.

Definition

A sequence $w \in \{A, B\}^{\mathbb{Z}}$ is *infinitely derivable* if it is admissible and all its derived sequences are still admissible.

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Characterization of Sturmian sequences

Theorem

A square cutting sequence is infinitely derivable.

Corollary

If w is a square cutting sequence with $\theta \in \Sigma_0$, the blocks of Bs have length n o $n+1$.

Example

The sequence $w = \dots ABBBBABBABBBBA \dots$ is NOT a square cutting sequence. Indeed: $w' = \dots ABBBABABBBA \dots$ and $w'' = \dots ABBAABBA \dots$ which is not admissible.

Theorem

Let w be infinitely derivable. Then w belongs to the closure of square cutting sequences.

Example

(infinitely derivable sequence which is not a square cutting sequence)
$\dots AAAAAABAAAAA \dots$

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Direction recognition and continued fractions

Let w a square cutting sequence, *for example:*
w = ...BBBABBABBABBABBBABBAABBBABBABBABBA...

Let us define $\{a_n\}_n \in \mathbb{N}^{\mathbb{N}}$ as follows:

- let a_0 such that the blocks of Bs in w have length a_0 or $a_0 + 1$ (*in the example* $a_0 = 2$: BB *o* BBB)
- let a_1 such that the blocks of As in $w^{(a_0)}$ (*in the example* w") have length a_1 or $a_1 + 1$ (*in the example* $a_1 = 3$: AAA *o* AAAA)
- ...
- let a_n such that the blocks of As (n odd) or Bs (n even) in $w^{(a_0+\cdots+a_{n-1})}$ have length a_n or $a_n + 1$;

Theorem (Direction Recognition)

The direction θ of the trajectory with cutting sequence w is given by:

$$\theta = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{\cdots + \cfrac{1}{a_n + \cdots}}}}$$

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let a_1 such that the blocks of As in $w^{(a_0)}$ (*in the example* w") have length a_1 or $a_1 + 1$ (*in the example* $a_1 = 3$: AAA *o* AAAA)

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Theorem (Direction Recognition)

The direction θ of the trajectory with cutting sequence w is given by:

$$\theta = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{\cdots + \cfrac{1}{a_n + \cdots}}}}$$

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Let w a square cutting sequence, *for example:*

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Proof for Sturmian sequences

The key step is given by the following Lemma:

Lemma

If w is a square cutting sequence, also the derived sequence w' is a square cutting sequence.

Recalling that a square cutting sequence is clearly admissible, we have:

Corollary

Square cutting sequences are infinitely derivable.

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If w is a cutting sequence of a trajectory in direction θ, the derived sequence w' is a cutting sequence of a trajectory in direction θ', where $\theta' = F(\theta)$ and F is the Farey map in Figure.

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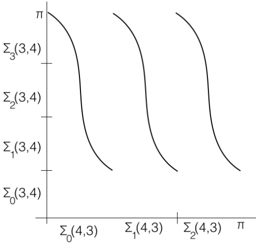

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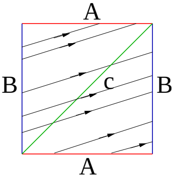

in the example:

w = . . . A B B   A B B B   A B B   A B B   A B B B   . . .

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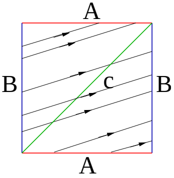

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Let us add the diagonal C.

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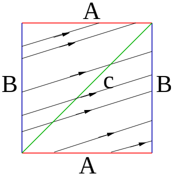

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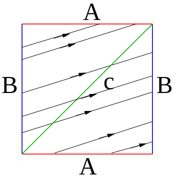

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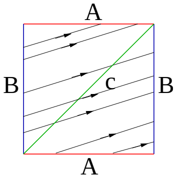

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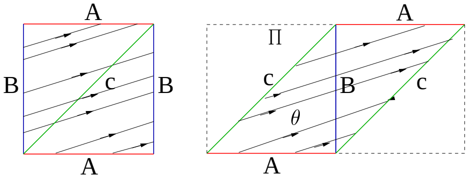

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Let us cut and paste the rectangle.

Consider the cutting sequence u with respect to the parallelogram Π.

To obtain u from $\tilde{w}$ it is enough to drop the Bs.

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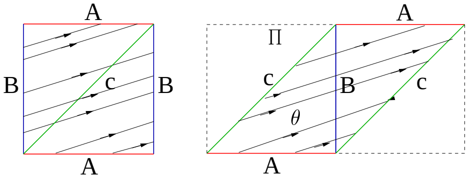

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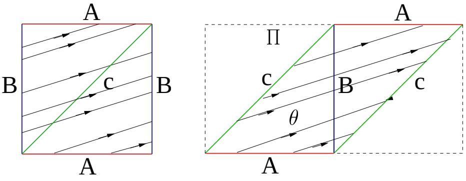

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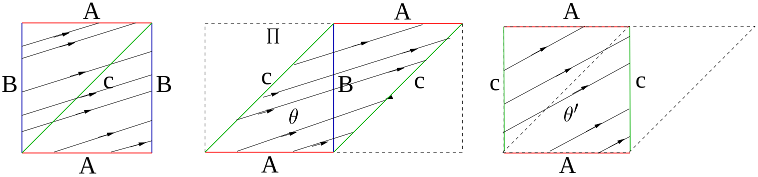

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u = ... A C A C C A C A C A C C ...

Let us *renormalize*: we can transform Π in a square by the shear $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Let us transform back the Cs into Bs.

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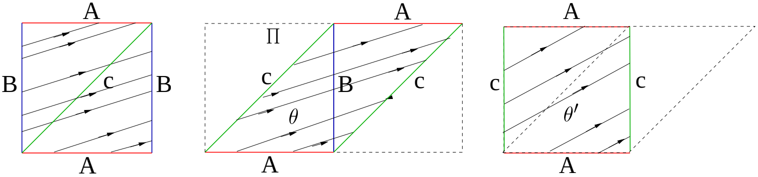

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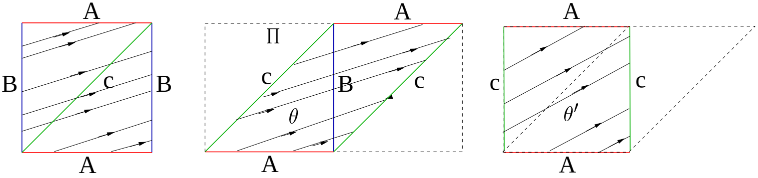

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The sequence thus obtained is the *derived sequence*.

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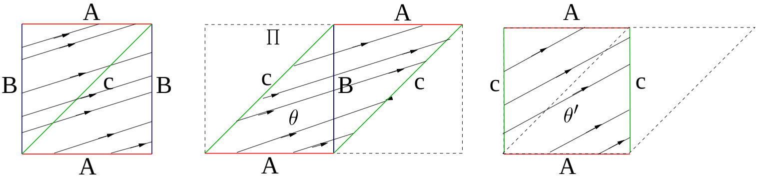

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To check it:

A BBB A → A BCBCB A → A CC A → A BB A

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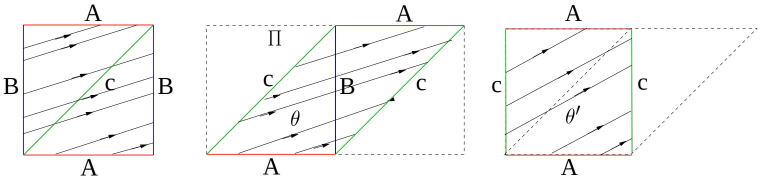

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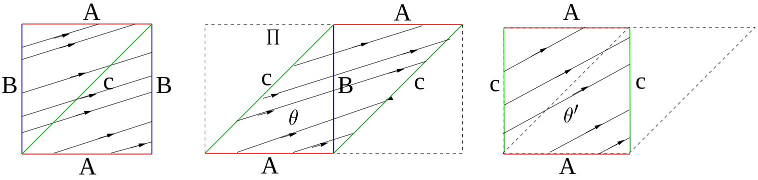

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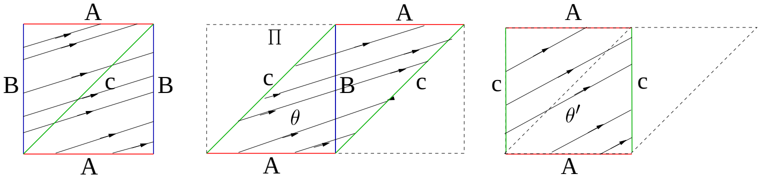

in the example:

$$
\begin{array}{llllllllllllllllllllllll}
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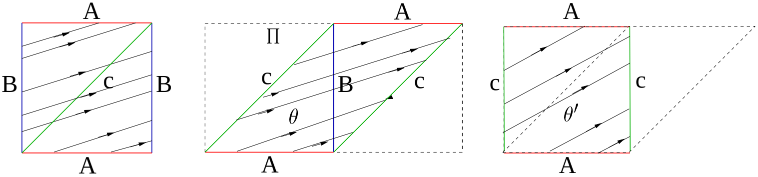

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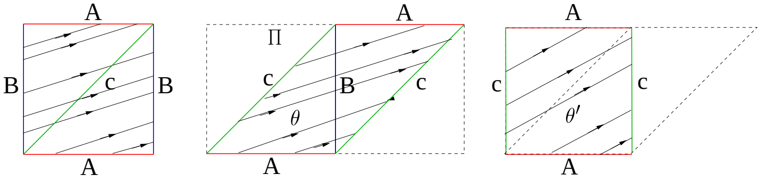

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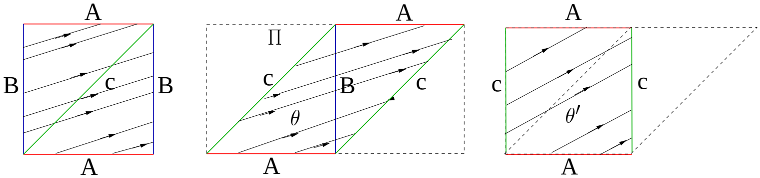

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