Beyond Sturmian: a characterization of octagon cutting sequences

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(based on joint work with John Smillie
Cornell University)

ICTP, Trieste, 23 July 2018
Linear trajectories and cutting sequences

Consider a regular polygon, for simplicity with $2n$ sides. As an example, in the talk we will consider a regular octagon. Glue opposite sides. Label pairs of sides by $\{A, B, C, D\}$.

Let $\varphi^\theta_t$ be the linear flow in direction $\theta$: trajectories which do not hit singularities are straight lines in direction $\theta$.

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The cutting sequence in $\{A, B, C, D\}^\mathbb{Z}$ that codes a bi-infinite linear trajectory of $\varphi^\theta_t$ consists of the sequence of labels of the sides hit by the trajectory.

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A classical case: Sturmian sequences

Consider the special case in which the polygon is a square.

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Sturmian sequences are characterized by having the smallest possible complexity among non-periodic sequences.

(Let $P_w(n)$ the number of words of length $n$ which appear in the sequence $w$: $P_w(n) = n$ iff $w$ is periodic. Sturmian sequences satisfy $P_w(n) = n + 1$.)
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In this case the cutting sequence correspond to the sequence of horizontal (letter A) and vertical (letter B) sides crossed by a line in direction $\theta$ in a square grid.
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Square cutting sequences are Sturmian sequences. They were studied since Hedlund and Morse.

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Motivation

Cutting sequences give a symbolic coding of the following systems:

- Translation surfaces
- Poligonal Billiards
- Interval exchange transformations
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  Glueing opposite sides one gets a surface of genus 2, with a flat metric with a singularity (it’s a translation surface); $\varphi_t^\theta$ is the geodesic flow with respect to the flat metric;

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For example: A billiard with angles $\frac{\pi}{2}$, $\frac{\pi}{8}$ and $\frac{3\pi}{8}$ (motion of a particle with elastic reflections at sides) by a procedure called unfolding is equivalent to the flow $\varphi^0_t$ in the octagon.

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- \textit{Interval exchange transformations}

The Poincaré first return map on a section is an interval exchange transformation (IET). As $\theta$ changes, one has a one-paramter family which is not generic.
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All these systems have entropy zero: cutting sequences have linear complexity.

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Translation surfaces and IETs which come from regular polygons are not generic: techniques that are used for the generic setting do not apply here.
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Characterization of cutting sequences

**Problem:**
Describe explicitly the symbolic sequences which are cutting sequences of trajectories.

In particular, answer the following questions:

D1) Which sequences in \( \{A, B, C, D\}^\mathbb{Z} \) are cutting sequences?

D2) Given a cutting sequence, can one recover the direction of the trajectory?
Given a finite piece of a cutting sequence, can one recover a sector of possible directions?

D3) Given a direction or more in general a sector of directions, can one produce all cutting sequences of trajectories in that direction or sector?
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  ▶ Characterization of Sturmian sequences;
     (revisiting Caroline Series work)
  ▶ Connection with Continued Fractions;
  ▶ Sketch of proof for the square;

▶ Regular polygons with $2n$ lati:

  *joint work with John Smillie (Cornell University)*
  ▶ Formulation of results in the case of the octagon;
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The octagon: isometries and sectors

Let $D_8$ be the isometries group of the octagon. The letters $\{A, B, C, D\}$ are invariant with respect to a central symmetry. The other elements induce permutations of $\{A, B, C, D\}$ for example:

$A \mapsto C$, $C \mapsto A$, $B \mapsto B$, $D \mapsto D$.

Let us assume that the direction of the trajectory is $\theta \in [0, \frac{\pi}{8}]$. A fundamental domain for $D_8$ is $\Sigma_0 := [0, \frac{\pi}{8}]$. Thus, acting by an element of $D_8$, up to a permutation of the letters $\{A, B, C, D\}$, we can consider $\theta \in \Sigma_0 := [0, \frac{\pi}{8}]$.

The other sectors of angle $\pi/8$ are in order $\Sigma_1, \Sigma_2, \ldots, \Sigma_7$. 
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The other sectors of angle $\pi/8$ are in order $\Sigma_1, \Sigma_2, \ldots, \Sigma_7$. 
The octagon: isometries and sectors

Let $D_8$ be the isometries group of the octagon. The letters $\{A, B, C, D\}$ are invariant with respect to a central symmetry. The other elements induce permutations of $\{A, B, C, D\}$ for example:

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The octagon: allowed transitions in $\Sigma_0$

Let $\theta \in \Sigma_0 := \left[0, \frac{\pi}{8}\right]$.

The transitions (pairs of consecutive letters) which can appear are:

Each octagon cutting sequence of a trajectory in direction $\theta \in \Sigma_0$ determines a path in the diagram in Figure.
The octagon: allowed transitions in $\Sigma_0$

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Each octagon cutting sequence of a trajectory in direction $\theta \in \Sigma_0$ determines a path in the diagram in Figure.
The octagon: possible transitions

Permuting the letters we obtain the diagrams corresponding to the other sectors:

- A → D → B → C
- C → B → D → A
- D → A → C → B
- B → C → A → D
- C → B → D → A
- B → A → C → D
- C → D → B → A
- A → B → D → C
Admissible sequences

Definition
A sequence $w \in \{A, B\}^\mathbb{Z}$ is *admissible* if it gives an infinite path on one of the following diagrams:

$D_0$

$D_1$

$D_2$

$D_3$

$D_4$

$D_5$

$D_6$

$D_7$

Lemma
An octagon cutting sequence is admissible.
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- $\mathcal{D}_1$: D \rightarrow A \rightarrow C \rightarrow B
- $\mathcal{D}_2$: D \rightarrow C \rightarrow A \rightarrow B
- $\mathcal{D}_3$: C \rightarrow D \rightarrow B \rightarrow A
- $\mathcal{D}_4$: C \rightarrow B \rightarrow D \rightarrow A
- $\mathcal{D}_5$: B \rightarrow C \rightarrow A \rightarrow D
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- $\mathcal{D}_7$: A \rightarrow B \rightarrow D \rightarrow C

Lemma
An octagon cutting sequence is admissible.
Derived sequences

Definition
A letter in \{A, B, C, D\} is *sandwiched* if it is preceded and followed by the same letter.

Example
In D B B C B A A D the letter C is *sandwiched* between to Bs.

Definition (Derived sequence)
If \( w \) is an octagon cutting sequence, the derived sequence \( w' \) is obtained erasing all letters which are *NOT sandwiched*.

Example
If \( w = \ldots D A D B C C B C C B D A D B C B D B D B C B D \ldots \),
\[ w' = \ldots A \]

Definition (Derivable sequences)
A sequence \( w \in \{A, B, C, D\}^\mathbb{Z} \) is *derivable* if it is admissible and its derivative is still admissible. The sequence \( w \) is *infinitely derivable* if each of its derivatives is derivable.
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**Definition**
A letter in \( \{A, B, C, D\} \) is *sandwitched* if it is preceeded and followed by the same letter.

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If \(w = \ldots D \underline{A} D B C C B C C B D A D B C B D B D B C B D \ldots\),
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In \( DBBCCBAAD \) the letter \( C \) is *sandwitched* between two \( B \)s.

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If \( w = \ldots D_{\text{A}} D_{\text{A}} D_{\text{B}} C_{\text{C}} C_{\text{B}} B_{\text{C}} C_{\text{B}} C_{\text{B}} D_{\text{A}} D_{\text{B}} C_{\text{B}} B_{\text{D}} B_{\text{D}} B_{\text{D}} B_{\text{D}} C_{\text{B}} B_{\text{D}} \ldots, \)
\( w' = \ldots A_{\text{A}} B_{\text{B}} A_{\text{A}} \)

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Necessary condition and sequence of sectors

**Theorem**

An octagon cutting sequence is infinitely derivable.

The converse is not true, but we can describe exactly the condition which one needs to add.

**Definition**

Let $w$ be infinitely derivable and let $w^{(n)}$ be the $n$th derived sequence. The sequence $\{s_k\}_{k \in \mathbb{N}} \in \{0, 1, \ldots, 7\}^\mathbb{N}$ is a sequence of sectors for $w$ if for each $k$, $w^{(k)}$ gives a path on the diagram $D_{s_k}$.

**Example**

\[
\begin{align*}
w &= \text{C C C B C C B D B} \\
&\quad s_0 = 0 \\
&\quad w' = \text{C B D B D} \\
&\quad s_1 = 4
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Theorem

An octagon cutting sequence is infinitly derivable.

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Octagon Continued Fractions

Let $F : [0, \pi] \rightarrow [\pi/8, \pi]$ the following map, that we call Octagon Farey map:

Definition
The octagon continued fraction expansion of $\theta$ is

$$\theta = [s_0, s_1, s_2, \ldots, s_k, \ldots] \text{ iff } \{\theta\} = \cap_k F_{s_0}^{-1}F_{s_1}^{-1} \ldots F_{s_k}^{-1}[0, \pi].$$

In this case we have $F^k(\theta) \in \Sigma_{s_k}$ per tutti i $k$. 
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Let $w$ be an octagon cutting sequence.

**Lemma**
If $w$ is not a periodic sequence, the sequence of sectors $\{s_k\}_{k \in \mathbb{N}}$ is univoquely determined. In particular, each derivative $w^{(k)}$ is admissible in an unique $\mathcal{D}_{s_k}$.

**Theorem**
If $w$ is not periodic, there is a unique sequence of sectors $\{s_k\}_{k \in \mathbb{N}}$ for $w$ and the direction of the trajectories with cutting sequence $w$ is given by

$$\theta = [s_0, s_1, s_2, \ldots]_O.$$
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**Ideas from proofs**

As in the case of the square, the theorems follow if we prove that:

**Theorem**

*If* \( w \) *is an octagon cutting sequence, also the derived sequence* \( w' \) *is an octagon cutting sequence.*

Furthermore, if \( w \) is the cutting sequence of a trajectory in direction \( \theta \), the derived sequence \( w' \) is a cutting sequence of a trajectory in direction \( \theta' = F(\theta) \), where \( F \) is the octagon Farey map.

To prove it, one uses an argument in the same spirit of:

i.e. one uses renormalization in the space of affine deformations of the octagon.

Regular polygons are rich of affine automorphism.
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Regular polygons are rich of affine automorphism.
Affine automorphism and Veech group

Let $S_O$ be the surface obtained glueing opposite sides of the octagon by translations: $S_O$ is an example of a translation surface, i.e. it has an atlas whose changes of coordinates are of the form $z \mapsto z + c$.

Definition

An automorphism $\Psi : S \mapsto S$ of a translation surface $S$ is an affine automorphism if it is affine in each chart and $D\Psi(z)$ is independent on $z \in S$. Let

$$\text{Aff}(S) = \{\psi, \quad \psi \text{ affine automorphism}\}.$$
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**Example (1)**

Se $\Psi \in D_8$ is an isometry of $O$, clearly one has $\Psi \in \text{Aff}(O)$. 
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**Example (2)**
The matrix \[
\begin{pmatrix}
1 & 2(1 + \sqrt{2}) \\
0 & 1
\end{pmatrix}
\] $\in V(O)$. 
The Veech group of the octagon

The Veech group \( V(S) \) is the group of linear parts of \( \text{Aff}(S) \):

\[
V(S) = \{ D\Psi, \quad \Psi \in \text{Aff}(S) \} \subset \text{SL}(2, \mathbb{R})^\pm.
\]

Ex 1

\[
V(T^2)^+ = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \rangle = \text{SL}(2, \mathbb{Z}).
\]

Ex 2

\[
V(O) = \langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 \sqrt{2} & 1 \sqrt{2} \\ -1 \sqrt{2} & 1 \sqrt{2} \end{pmatrix}, \begin{pmatrix} 1 & 2(1 + \sqrt{2}) \\ 0 & 1 \end{pmatrix} \rangle
\]

If \( S \) is a translation surface glued from a regular polygon, \( V(S) \) is a lattice in \( \text{SL}(2, \mathbb{R})^\pm \) (Veech)

The surfaces \( S \) for which \( V(S) \) is a lattice are actively researched in Teichmüller dynamics.
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**Ex 1**

$$V(\mathbb{T}^2)^+ = < \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} > = \text{SL}(2, \mathbb{Z}).$$

**Ex 2**

$$V(O) = < \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \begin{pmatrix} 1 & 2(1 + \sqrt{2}) \\ 0 & 1 \end{pmatrix} >$$

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$$V(S) = \{ D\psi, \quad \psi \in \text{Aff}(S) \} \subset \text{SL}(2, \mathbb{R})^\pm.$$ 

**Ex 1**

$$V(\mathbb{T}^2)^+ = < \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} > = \text{SL}(2, \mathbb{Z}).$$

**Ex 2**

$$V(O) = < \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 1 & 2(1 + \sqrt{2}) \\ 0 & 1 \end{pmatrix} >$$

If $S$ is a translation surface glued from a regular polygon, $V(S)$ is a lattice in $\text{SL}(2, \mathbb{R})^\pm$ (Veech).

The surfaces $S$ for which $V(S)$ is a lattice are actively researched in Teichmüller dynamics.
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Let $w$ be the cutting sequence of a trajectory in direction $\theta \in \Sigma_0$. Let $O' = \begin{pmatrix} -1 & 2(1 + \sqrt{2}) \\ 0 & 1 \end{pmatrix} O$.

**Lemma**

The derived sequence $w'$ coincides with the cutting sequence of the same trajectory in direction $\theta$ with respect to the sides of $O'$.

Let us renormalize:

$O' \mapsto O$

$\theta \mapsto \theta'$

**Lemma**

The derived sequence $w'$ is an octagon cutting sequence in direction $\theta' = F_O(\theta)$.
Derivation and renormalization in the octagon

Let \( w \) be the cutting sequence of a trajectory in direction \( \theta \in \Sigma_0 \). Let
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Renormalization, modular surface and continued fractions

The space of lattices is $\mathbb{H}/SL(2, \mathbb{Z})$. (moduli space of tori with a flat metric)

Farey Tessellation: $\mathcal{V}(Q) = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \rangle \subset \mathcal{F}(0, 1, \infty)$

$Q$ square

$S_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$S_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

Given $\theta$, $g_t^\theta$ geodesics

$\theta = \frac{1}{a_0 + \frac{1}{a_1 + \ldots}}$

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Renormalization and dynamics on the Teichmüller disk

\[ SL(2, \mathbb{R}) \cdot O = \{ A \cdot O, A \in SL(2, \mathbb{R}) \} \] affine deformations of the octagon

\[
\begin{align*}
\text{affine deformations} & \quad = \quad \frac{SL(2, \mathbb{R})}{V(O)} \\
\text{affini automorphisms} & \quad = \quad \text{(Teichmueller disk)}
\end{align*}
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\[ V(O) \cdot O \] centers of ideal octagons
tree of renormalization moves
Given \( \theta, g^\theta_t \) geodesics is approximated by a sequence of renormalization moves
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Generation of cutting sequences

Let us define operators $g^j_i$ that invert derivation.
The operator $g^j_i$ interpolates a sequence $w$ admissible in $\mathcal{D}_j$ producing a sequence ammissible in $\mathcal{D}_i$ and such that $(g^j_i(w))' = w$.

Definition
Let $w$ be ammissible in $\mathcal{D}_0$. The sequence $g^2_0 w$ us obtained interpolating the letters corresponding to vertices in Figure with the words on the arrows:

Example

$w = \ldots \text{D C A B B B A C} \ldots$
$g^3_0 w = \ldots \text{D B C} \ldots$

The only sandwiched letters are the coloured ones, thus $(g^3_0 w)' = w$. 

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![Diagram](diagram.png)

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```
D B C A CC
B BD CCBD
DBCCBD
```

**Example**

\[
\begin{align*}
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g^3_0 w & = \ldots \ D \ B \ C \ldots 
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**Definition**

Let $w$ be admissible in $\mathcal{D}_0$. The sequence $g_0^2w$ is obtained interpolating the letters corresponding to vertices in Figure with the words on the arrows:

Example

\[
\begin{align*}
  w &= \ldots \, D \, C \, A \, B \, B \, B \, A \, C \, \ldots \\
  g_0^3w &= \ldots \, D \, B \, C \, BD \, A \, \ldots 
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![Diagram](image)

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$w = \ldots$ D C A B B B A C $\ldots$

$g_0^3w = \ldots$ D B C BD A DBCC B $\ldots$

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<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th>BD</th>
<th>DBCC</th>
<th>CCBD</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>C</td>
<td>A</td>
<td>B</td>
<td></td>
</tr>
<tr>
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\begin{align*}
w &= \ldots \text{D} \quad \text{C} \quad \text{A} \quad \text{B} \quad \text{B} \quad \text{B} \quad \text{A} \quad \text{C} \quad \ldots \\
g^3_0 w &= \ldots \text{D} \quad \text{B} \quad \text{C} \quad \text{BD} \quad \text{A} \quad \text{DBCC} \quad \text{B} \quad \text{CC} \quad \text{B} \quad \text{CC} \quad \text{B} \quad \text{CCBD} \quad \text{A} \quad \text{DB} \quad \text{C} \quad \ldots \end{align*}
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The interpolation operators

The other operators are obtained from these ones by permuting the letters.
Characterization of the closure of cutting sequences

Lemma

If \( w \) is a cutting sequence and \( \{s_n\}_{n \in \mathbb{N}} \) a sequence of sectors, we have

\[
w \in \bigcap_n g_{s_1}^{s_0} \ldots g_{s_n}^{s_{n-1}} \{A, B, C, D\}^\mathbb{Z}.
\]

This condition, together with infinite derivability, is necessary and sufficient to characterize the closure of octagon cutting sequences (in \( \{A, B, C, D\}^\mathbb{Z} \)):

Theorem (Smillie-U)

A sequence \( w \in \{A, B, C, D\}^\mathbb{Z} \) belongs to the closure of cutting sequences in the octagon iff there exists a sequence \( \{s_n\}_{n \in \mathbb{N}} \in \{0, \ldots, 7\}^\mathbb{N} \) such that

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An analogous theorem holds for every regular polygon.
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