# Introduction to Dynamical Systems
# Lecture Notes

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# Contents

# Part I

# Basic concepts and examples

# Chapter 1

# Maps

The general formal definition of a Dynamical System will be given below in terms of the somewhat intimidating notion of a “semi-group of transformations”, but the simplest example of such a structure is extremely natural and easy to understand and therefore we begin by illustrating the basic notions and fundamental examples in such a simpler setting. Many definitions can be formulated starting from an arbitrary set $X$ and an arbitrary map

$$f : X \to X$$

defined on $X$. This setup can be considered the most basic form of a “model” of a real-life process where $X$ denotes the *phase space*, i.e. the set of all possible states of a given system, and $f$ denotes the “physics” of the process, i.e. the law that determines the evolution of the system.

## 1.1 Fundamental definitions

Starting from the framework described above we can make several important observations and definitions.

### 1.1.1 Initial conditions and orbits

For each given state or initial condition $x_0 \in X$ the map $f$ gives us the the new state of the system

$$x_1 := f(x_0)$$

after one unit of time or one application of the process. Since $x_1 \in X$ we can apply the map $f$ again to this new state, and obtain the state $x_2 = f(x_1)$ which gives the state of the system starting from $x_0$ after two units of time or applications of the process. More generally we can *iterate* the map any number of times and it is therefore useful to define formally the $n$'th iterate of $f$ by

$$f^n := f \circ \cdots \circ f$$

where $\circ$ denotes the composition of maps. Then, given an initial condition $x_0 \in X$, the $n$'th *image* of $x_0$ under $n$ iterations of the map is given by

$$x_n := f^n(x_0).$$

With this notation we make the following

**Definition 1.1.** The (forward) orbit of $x_0 \in X$ is the set

$$\mathcal{O}^+(x_0) := \{x_n\}_{n\in\mathbb{N}}$$

Recall that if the map $f : X \to X$ is bijective, then is is *invertible* in the sense that the inverse map

$$f^{-1} : X \to X$$

is well defined by the condition $f^{-1} \circ f = f \circ f^{-1} =$*identity.* This can be intuitively thought of as going "backwards" in time, if $x_1 = f(x_0)$ is the evolution of the initial condition $x_0$ under one application of $f$, then applying the inverse map to $x_1$ we get $f^{-1}(x_1) = f^{-1}(f(x_0)) = f^{-1} \circ f(x_0) = x_0$ so that we recover the initial state, i.e. recover what happened in the past.The inverse $f^{-1}$ is itself a map to all effects and thus can itself be iterated, and we let

$$f^{-n} := f^{-1} \circ \cdots \circ f^{-1}$$

denote the $n$'th fold composition of $f^{-1}$. Then, we can define the sequence of *pre-images* of some initial condition $x_0 \in X$ under $n$ iterations of the map $f$ as

$$x_{-n} := f^{-n}(x_0).$$

Thus if $f$ is invertible we can consider both forward time and backward time iterates of $f$ and make the following

**Definition 1.2.** If $f$ is invertible, the *full orbit* of $x_0$ is the set

$$\mathcal{O}(x_0) = \{x_n\}_{n\in\mathbb{Z}}$$

*Remark* 1.3. If $f$ is not one-to-one because then several initial conditions might map to the same point $x_1$ and thus $x_1$ may have several possible pasts and $f^{-1}$ cannot be defined.

If $f$ is not invertible it can still be useful to consider the set

$$f^{-n}(x_0) := \{y \in X : f^n(y) = x_0\}$$

of *pre-images* of $x_0$. In general this may contain lots of points and this we cannot really talk about the set of such preimages forming an orbit of the point $x_0$.

The basic goal of the theory if Dynamical Systems is essentially to *describe* the orbits associated to the map $f$, including how they depend on the initial condition and possibly how they change if the map $f$ is slightly perturbed. The possibility of achieving this goal often relies on significant additional structure on the set $X$ and on the map $f$ and the introduction of remarkably complex and sophisticated ideas. We will introduce these additional structures and ideas as they are required in the course of our study.

### 1.1.2 Fixed and periodic points

The simplest orbits are those associated to fixed points.

**Definition 1.4.** $x_0$ is a *fixed point* if $f(x_0) = x_0$.

If $x_0$ is a fixed point then clearly $x_1 := f(x_0) = x_0$ and thus $x_n = x_0$ for all $n \in \mathbb{N}$. In particular, we have $\mathcal{O}^+(x) = \{x_0\}$, i.e. the forward orbit of $x_0$ simply consists of the point $x_0$ itself. If $f$ is invertible and $x_0$ is a fixed point for $f$ then it is also a fixed point for $f^{-1}$ (Exercise **??**) and therefore the full forward and backward orbit reduces to the single point $x_0$ itself: $\mathcal{O}(x) = \{x_0\}$.

A generalization of the concept of a fixed point is that of a *periodic* point.

**Definition 1.5.** $x_0$ is a *periodic point* of period $k \geq 1$ if $f^k(x_0) = x_0$.

Notice that a fixed point is also periodic of every period $k \geq 1$. More generally, if $x_0$ is a periodic point of period $k$ then it is also a periodic orbit of period $nk$ for any integer $n \geq 1$ since $f^{nk}$ can be written as the $n$'fold composition of $f^k$, i.e. $f^{nk} = f^k \circ \cdots \circ f^k$. Thus it is useful to define the notion of minimal period of $x_0$.

**Definition 1.6.** $x_0$ has *minimal* period $k$ if $k = \min\{k \geq 1 : f^k(x_0) = x_0\}$.

If $x_0$ is a periodic point of minimal period $k \geq 1$, the forward orbit of $x_0$ is the finite set $\mathcal{O}^+(x_0) = \{x_0, ..., x_{k-1}\}$. To conclude this section we remark that a point may not be periodic but may at some point "land" on a periodic point.

**Definition 1.7.** $x_0$ is *pre-periodic* if $x_j$ is periodic for some $j \geq 0$.

Notice that, formally, periodic points are also pre-periodic but the converse is of course not true. If $x_0$ is pre-periodic, its orbit is the finite set $\mathcal{O}^+(x_0) = \{x_0, ..., x_j, ..., x_{j+k-1}\}$. where $x_0, ..., x_{j-1}$ is sometimes referred to as the *transient* part of the orbit, and $x_j, ..., x_{j+k-1}$ is the *periodic* part. It is easy to see that any point whose orbit $\mathcal{O}^+(x_0)$ is finite must be periodic or pre-periodic (Exercise **??**). Moreover, true pre-periodic points can only occur for non-invertible maps, since if $f$ is invertible any pre-periodic point must be periodic (Exercise **??**).

### 1.1.3 Limit sets

Periodic and pre-periodic orbits (including those associated to fixed point) are pretty much as far as we can go in the description of the orbits of a map $f$ without adding any additional structure to the set $X$. Indeed, by Exercise **??** any orbit which is not periodic or pre-periodic is infinite and there is not much we can say to "describe" an infinite subset of an arbitrary set $X$ with no additional structure. There are many kinds of structures that can be considered on a set, e.g. measure-theoretic structure, topological structure, geometric structure, algebraic structure. Each of these could be more or less useful depending on the kind of description and information we are interested in and the kind of map $f$ which we are considering. The kind of structure that the set $X$ admits gives rise to its own set of questions and methods and problems and solutions and essentially to a distinct approach and branch of the theory of Dynamical Systems, although naturally there are also many situations in which $X$ may admit a multiplicity of structures and these can all contribute to a deeper understanding of the system.

In these notes we will mainly focus on the *topological* properties of Dynamical Systems and thus suppose from now on that $X$ is a *topological space*. In some situations, particularly for specific examples, we will often have additional structures, such as a metric space structure, or even a geometric structure, but the general point of view will be to concentrate on the topological structure and the properties of the dynamics that can be described through that. The key, and in some sense only, feature of a topological structure is to allow us to define the notion of limit points for sequences and we take full advantage of this in our quest to describe the structure of non-periodic orbits.

**Definition 1.8.** Let $X$ be a topological space and $f : X \to X$ a map. For $x_0 \in X$, we define the *omega-limit* set $\omega(x_0)$ of $x_0$ as

$$\omega(x_0) := \{y \in X : x_{n_j} \to y \text{ for some subsequence } n_j \to \infty \text{ as } j \to \infty\}.$$

The omega-limit is thus the set of topological limit points of the forward orbit $\mathcal{O}^+(x_0)$ considered as an infinite sequence in $X$. It is easy to see that if $x_0$ is periodic, then $\omega(x_0) = \mathcal{O}^+(x_0)$ (Exercise 1.4.1). Moreover, is $X$ is compact then $\omega(x_0) \neq \emptyset$ for every $x_0 \in X$ (Exercise 1.4.2 and, if $f$ is continuous, then $\omega(x_0)$ is *$f$-invariant* in the sense that $x \in \omega(x_0)$ implies $f(x) \in \omega(x_0)$ (Exercise **??**) If $f$ is invertible we can define similarly the *alpha-limit* set:

$$\alpha(x_0) := \{x \in X : x_{n_j} \to x \text{ for some subsequence } n_j \to -\infty \text{ as } j \to \infty\}.$$

Clearly the same properties apply to the alpha-limit set that apply to the omega-limit set. The notion of $\alpha$ and $\omega$ limit set can be used to formulate two notions which we will use below: that of an attractor and of a repeller. The alpha limit set is simply the omega limit for $f^{-1}$ and thus all the same properties apply.

The omega and alpha limits sets depend of course on the space $X$ and on the map $f$ as well as the initial condition $x_0$. A priori they can be anything, from a single point to the whole space, as we shall see from several of the examples to be studied below. Indeed, these two cases are the most interesting and we give some associated definitions and properties which hold in general. First of all, following standard terminology in topology, we say that the orbit $\mathcal{O}^+(x_0)$ is *dense* in $X$ if all points of $X$ are limit points of the sequence $\mathcal{O}^+(x_0)$, i.e. if $\omega(x_0) = X$. The existence of a dense orbit is a non-trivial and very relevant property and we this formulate the following notion.

**Definition 1.9.** $f : X \to X$ is *transitive* if there exists $x_0 \in X$ with $\omega(x_0) = X$.

On the other extreme, we have situations where the orbit of some point $x_0$ has a single point as its omega-limit. In this case we have the following

**Lemma 1.10.** *Let $f : X \to X$ be a continuous map on a topological space. Suppose there exist points $x_0, p \in X$ such that $\omega(x_0) = \{p\}$. Then $p$ is a fixed point.*

*Proof.* Exercise 1.4.1. □

**Definition 1.11.** The point $p \in X$ is an *attracting fixed point* if there exists a neighbourhood $\mathcal{U}$ of $p$ such that $\omega(x_0) = \{p\}$ for all $x \in \mathcal{U}$.

If $f$ is invertible we define a *repelling fixed point* $p$ as an attracting fixed point for $f^{-1}$ or, equivalently, as a point for which there exists a neighbourhood $\mathcal{U}$ of $p$ such that $\alpha(x_0) = \{p\}$ for all $x \in \mathcal{U}$.

## 1.2 Fundamental examples

We now give easy and fundamental examples of dynamical systems which help us to illustrate the notions defined above and also serve as models for more general systems.

### 1.2.1 Contracting maps

Let $X = [0, 1]$, $\lambda \in (0, 1)$ and consider the map $f : X \to X$ defined by

$$f(x) = \lambda x.$$

Then it is very easy to see that for every $x_0 \in X$, we have $x_1 = \lambda x_0$ and so $x_2 = \lambda x_1 = \lambda^2 x_0$ and, more generally, for every $n \geq 1$ we have

$$x_n = \lambda^n x_0.$$

Since $\lambda \in (0,1)$ we have that $\lambda^n \to 0$ as $n \to \infty$ and therefore $x_n \to 0$ as $n \to \infty$ and therefore the orbit of $x_0$ *converges* to 0 as $n \to \infty$ and thus in particular has a unique limit point, which is precisely 0. Therefore $\omega(x_0) = 0$ for *every* initial condition $x_0 \in X$. This shows that according to Definition 1.11, $p = 0$ is an attracting fixed point. In fact this is a special case of an attracting fixed point because we can take $\mathcal{U} = X$. In this case we say that the attracting fixed point $p$ is a *globally attracting fixed point*. In fact this situation is relatively common and occurs under some general conditions.

**Definition 1.12.** Let $(X,d)$ be a metric space.

1. A sequence $x_1, x_2, x_3, ...$ in $X$ is *Cauchy* if for every $\epsilon > 0$ there exists an $N$ such that for all $m, n > N$, $d(x_n, x_m) < \epsilon$.
2. $X$ *complete* if every Cauchy sequence in $X$ converges to an element of $X$.
3. $f : X \to X$ is a *contraction* if there exists a constant $\lambda \in (0,1)$ such that

$$d(f(x), f(y)) \leq c\, d(x,y) \tag{1.1}$$

for all $x, y \in X$.

The following is a standard and very useful result with many applications.

**Theorem 1.13** (Contraction Mapping)**.** *Let $(X,d)$ be a complete metric space and $f : X \to X$ a contraction. Then*

*(i) $f$ has a unique fixed point $p \in X$.*
*(ii) For any $x_0 \in X$ the sequence $\{f^n(x)\}$ converges to $p$.*

*Proof.* Exercises 1.4.7-1.4.9. □

### 1.2.2 Translations and circle rotations

Let $X = \mathbb{S}^1$ be the unit circle which we can define as the quotient space $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ where $x \sim y$ if $|x - y| = 1$ or equivalently just as the unit interval $[0,1]/\sim$ with the identification $0 \sim 1$. Then, if $x \in \mathbb{R}$ is any real number, it's non-integer part, $x \mod 1$, belongs to $\mathbb{S}^1$.

**Definition 1.14.** $f : \mathbb{S}^1 \to \mathbb{S}^1$ is a *rotation* or a *translation* by $\theta$ if

$$f(x) = x + \theta \mod 1.$$

**Proposition 1.15.** *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a rotation by $\theta$. Then*

*1. $\theta$ is rational if and only if every orbit is periodic;*
*2. $\theta$ is irrational if and only if every orbit is dense in $\mathbb{S}^1$.*

*Proof.* Exercise 1.4.3. □

Circle rotations are very special in several respects, not least of which the fact that, whether $\theta$ is rational or irrational, all points have the same behaviour, either they are all periodic or they are all dense.

### 1.2.3 Expanding maps

We start by looking at a relatively simple example of a very concrete map which however is the archetypal example of a very large class of important systems. Indeed, consider the map $f : \mathbb{S}^1 \to \mathbb{S}^1$ defined by

$$f(x) = 10x \mod 1.$$

This map is very easy to study directly by considering all numbers in $\mathbb{S}^1$ in their decimal representation. Recall that every real number has an *infinite* decimal representation (possibly ending in an infinite number of zeroes) and that mod 1 in the definition of $f$ means that we only consider the non-integer part of the number $10x$. Suppose for example that $x_0 = 0.2743959....$ Then it is easy to see that $x_1 = f(x_0) = 0.743959...$, $x_2 = f(x_1) = 0.43959...$, $x_3 = 0.3959....$ From this it is easy to deduce several elementary properties of the dynamics, such as the number of fixed points and periodic points of any period, the existence of dense orbits, etc. (Exercise 1.4.4).

We chose the example $f(x) = 10x$ mod 1 because multiplication by 10 works particularly nicely with the decimal representation of numbers and allows us to verify explicitly the properties discussed above. However, we can also let $\kappa \geq 2$ be an integer and define the map

$$f(x) = \kappa x \mod 1$$

Then it is (almost) just as easy to see that such a map also has periodic points of any period, dense orbits, etc (Exercise 1.4.6).

**Definition 1.16.** Let $X$ be a metric space. A map $f : X \to X$ is said to exhibit *sensitive dependence on initial conditions* if there exists $\epsilon > 0$ such that for every $x \in X$ and every $\delta > 0$ there exists $y \in X$ and $n \geq 1$ with $d(x, y) \leq \delta$ and $d(f^n(x), f^n(y)) \geq \epsilon$.

The sense of this definition is that no matter any arbitrarily small "mistake" in your choice of initial condition, i.e. choosing $y$ instead of $x$, eventually leads to a macroscopic difference in outcomes. It is easy to check that contractions and circle rotations do not exhibit this property, whereas maps of the form $f(x) = \kappa x$ mod 1 do (Exercise 1.4.5).

## 1.3 Discrete Time Dynamical Systems

Given an arbitrary set $X$ and an arbitrary map $f : X \to X$, we consider the family

$$\{f^t\}_{t\in\mathbb{N}} \tag{1.2}$$

of all (forward) iterates of $f$. By convention we let $f^0 = Id$ denote the identity map. It is then easy to see that the following properties hold:

**Closure:** $f^s \circ f^t = f^{s+t}$ for all $s, t \in \mathbb{N}$;
**Identity:** $f^0 = Id$;
**Associativity:** $f^r \circ (f^s \circ f^t) = (f^r \circ f^s) \circ f^t$ for all $r, s, t \in \mathbb{N}$.

Therefore (1.2) is a *semi-group* of maps under composition. If $f$ is invertible then we can consider the family

$$\{f^t\}_{t\in\mathbb{Z}} \tag{1.3}$$

which satisfies:

**Closure:** $f^s \circ f^t = f^{s+t}$ for all $s, t \in \mathbb{N}$;
**Identity:** $f^0 = Id$;
**Associativity:** $f^r \circ (f^s \circ f^t) = (f^r \circ f^s) \circ f^t$ for all $r, s, t \in \mathbb{N}$;
**Inverse:** For every $s \in \mathbb{G}$ there exists $t \in \mathbb{G}$ such that $f^s \circ f^t = f^s \circ f^t = Id$.

The family (1.3) is therefore a *group* of transformations under composition.

Notice moreover that if $X$ has some additional structure and $f$ has some regularity such as being continuous or differentiable (in the non-invertible case) or if $f$ is a homeomorphism or a diffeomorphisms (in the invertible case), then the same is true for the compositions of these maps, In which case the families (1.2) and (1.3) are groups or semi-groups of transformations all of which have the same regularity.

## 1.4 Exercises

### 1.4.1 Limits sets

**Exercise 1.4.1.** Suppose that $X$ is a metric space and let $f : X \to X$.

1. Show that if $\mathcal{O}^+(x_0)$ is a finite set, then $x_0$ is periodic or pre-periodic.
2. Show that if $f$ is invertible, then any pre-periodic orbit is periodic.
3. Show that if $x_0$ is a periodic point, then $\omega(x_0) = \mathcal{O}^+(x_0)$
4. Suppose $x_0$ is a pre-periodic point. What is $\omega(x_0)$ ?

**Exercise 1.4.2.** Suppose that $X$ is a metric space and let $f : X \to X$.

1. Show that if $X$ is compact then $\omega(x_0) \neq \emptyset$ for any initial condition $x_0 \in X$.
2. Show that if $f : X \to X$ is continuous then for any initial condition $x_0 \in X$ the set $\omega(x_0)$ is forward invariant, i.e. if $x \in \omega(x_0)$ then $f(x) \in \omega(x_0)$.
3. Let $f : X \to X$ be a continuous map on a metric space and let $x_0 \in X$. Suppose there exist points $x_0, p \in X$ such that $\omega(x_0) = \{p\}$. Let $x_1 = f(x_0)$. (a) Show that $\mathcal{O}^+(x_1) = f(\mathcal{O}^+(x_0))$. (b) Deduce that $\omega(x_1) = \omega(x_0) = \{p\}$. (c) Using the continuity of $f$, show that $p$ is a fixed point.

### 1.4.2 Circle Rotations

**Exercise 1.4.3.** Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a circle rotation by some angle $\theta$.

1. Suppose $\theta = p/q$ where $p, q \in \mathbb{N}$. Compute explicitly the form of the iterates $f^n$ for $n \geq 1$ and deduce that every point is periodic of period $q$. Conversely, assume that there exists some periodic point $x$ of some period $q$. Deduce that $q\theta = 0 \mod 1$ and therefore that $\theta$ is rational.
2. Suppose there exists $x_0$ with a dense orbit. Deduce that there cannot be *any* periodic orbit and therefore $\theta$ must be irrational.
3. Suppose $\theta$ is irrational and assume first that $\theta > 0$. Fix an arbitrary $\epsilon > 0$ and cover $\mathbb{S}^1$ with a finite number of arcs of length $\leq \epsilon$.
   (a) Explain first why it is sufficient to prove that the orbit of every initial condition $x_0$ intersects each of these arcs, to imply that every orbit is dense.
   (b) To show that the orbit of $x_0$ intersects each of these arcs, show first that there must be at least one arc that contains at least two points $x_m, x_n \in \mathcal{O}(x_0)$.
   (c) Deduce that there exist some $\ell \in \mathbb{Z}$ such that $f^\ell$ is a circle rotation by some angle $\leq \epsilon$.
   (d) Conclude that the orbit of $x_0$ intersects every arc.

### 1.4.3 Expanding maps

**Exercise 1.4.4.** Let $f(x) = 10x \mod 1$.

1. Give an example of a fixed point and of periodic points of period 2 and 3. Show how to give examples of periodic points of any given minimal period.
2. Show that the set $\text{Per}(f)$ of periodic points of $f$ is dense in $\mathbb{S}^1$.
3. Let $p = 1/3 = 0.33333.....$ Show that there exists a point $x_0$ which is *not* fixed, periodic, or pre-periodic, such that $p \in \omega(x_0)$.
4. Let $q = 2/6 = 0.66666....$ Show that there exists a point $x_0$ which is *not* fixed, periodic, or pre-periodic, such that $p \in \omega(x_0)$ *and* $q \in \omega(x_0)$.
5. Show that there exists a point $x_0$ such that $\omega(x_0) = \mathbb{S}^1$.

**Exercise 1.4.5.** (a) Show that contraction mappings and circle rotations do not exhibit sensitive dependence on initial conditions. (b) Show that maps of the form $f(x) = \kappa x \mod 1$, for some integer $\kappa \geq 2$, exhibit sensitive dependence on initial conditions.

**Exercise 1.4.6.** Let $f(x) = \kappa x \mod 1$ for some integer $\kappa \geq 2$. Show that

1. the set $Per(f)$ of periodic points of $f$ is dense in $\mathbb{S}^1$;
2. there exists a point $x_0$ such that $\omega(x_0) = \mathbb{S}^1$.

### 1.4.4 Contraction Mapping Theorem

The final three exercises prove the Contraction Mapping Theorem. We suppose that $(X, d)$ is a complete metric space and $f : X \to X$ is a contraction as per the assumptions of Theorem 9.6.

**Exercise 1.4.7.** Suppose $a$ and $b$ are distinct fixed points of $f$. Show that this contradicts the contraction property (1.1). This shows that there can be *at most* one fixed point, and thus proves uniqueness.

Since $X$ is complete, if we show that the sequences of iterates given in $(ii)$ is Cauchy, then we know it converges to a point in $X$.

**Exercise 1.4.8.** a. From the construction of the sequence we have

$$d(x_n, x_{n+1}) = d(f(x_{n-1}), f(x_n)) \leq c\, d(x_{n-1}, x_n).$$

Using this, find a bound for $d(x_n, x_{n+1})$ in terms of $c, x_0$, and $x_1$.

b. Using the method in a., for $m > n$, find a bound for $d(x_n, x_m)$ in terms of $c, x_0$, and $x_1$.

c. Prove that the sequence of iterates is a Cauchy sequence. Since $X$ is complete, then the sequence $\{x_n\}$ converges in $X$.

**Exercise 1.4.9.** a. Prove that $f$ is continuous.

b. Let $a = \lim_{n\to\infty} x_n$. Show that the continuity of $f$ implies

$$\lim_{n\to\infty} f(x_n) = f(a).$$

c. Deduce that $a$ is a fixed point of $f$.

# Chapter 2

# Flows

## 2.1 Continuous Time Dynamical System

The notion of a Dynamical System as a semi-group or group of transformations of a set $X$ leads to a natural generalization using *continuous time*. For this we need a little more structure on the space $X$ and we therefore assume from now on that $X$ is a metric space.

### 2.1.1 Formal definition

Let

$$\{f^t\}_{t\in\mathbb{R}^+} \tag{2.1}$$

be a family of maps $f^t : X \to X$ which depends *continuously* on the parameter $t \in \mathbb{R}^+$, and which forms a *semi-group* of transformations of $x$ under composition, i.e. satisfies the following properties

**Closure:** $f^s \circ f^t = f^{s+t}$ for all $s, t \in \mathbb{R}^+$;
**Identity:** $f^0 = Id$;
**Associativity:** $f^r \circ (f^s \circ f^t) = (f^r \circ f^s) \circ f^t$ for all $r, s, t \in \mathbb{R}^+$.

Then we say that (2.1) is (non-invertible) dynamical system in *continuous time*, or a *semi-flow* on $X$. Let

$$\{f^t\}_{t\in\mathbb{R}} \tag{2.2}$$

be a family of maps $f^t : X \to X$ which depends *continuously* on the parameter $t \in \mathbb{R}$, and which forms a *group* of transformations of $x$ under composition, i.e. satisfies the following properties

**Closure:** $f^s \circ f^t = f^{s+t}$ for all $s, t \in \mathbb{R}$;
**Identity:** $f^0 = Id$;
**Associativity:** $f^r \circ (f^s \circ f^t) = (f^r \circ f^s) \circ f^t$ for all $r, s, t \in \mathbb{R}$;

**Inverse:** For every $s \in \mathbb{R}$ there exists $t \in \mathbb{R}$ such that $f^s \circ f^t = f^s \circ f^t = Id$.

Then we say that (2.2) is an (invertible) Dynamical System in *continuous time*, or a *flow* on $X$.

### 2.1.2 Basic concepts

Notice that we can generalize in obvious ways all the basic definitions given above for discrete time dynamical systems. In particular we can define the forward orbit of a point $x$ and, in the case of a flow, the full orbit of a point $x$ as the sets

$$\mathcal{O}^+(x) := \{f^t(x)\}_{t\in\mathbb{R}^+} \quad \text{and} \quad \mathcal{O}(x) := \{f^t(x)\}_{t\in\mathbb{R}}$$

respectively, with the obvious difference that the orbit is no longer in general a countable set. Indeed, notice that by the assumption on the continuity of the family of maps with respect to the parameter it follows that the orbit $\mathcal{O}^+(x)$ or $\mathcal{O}(x)$ are *continuous images* of $\mathbb{R}^+$ or $\mathbb{R}$ respectively in $X$. In the case of a fixed point we have $f^t(x) = x$ for all $t$, in which case the orbit reduces to the single point $x$, but for a periodic point there exists some $T > 0$ such that $f^t(x) \neq x$ for all $t \in (0, T)$ and $f^T(x) = x$ (in the continuous time case it is convenient to distinguish a fixed point from a non-trivial periodic orbit since one consists of a single point whereas the other consists of an uncountable set of points). Similarly we can also define $\omega$-limit sets and, for flows, $\alpha$-limit sets, as limit points of the forward and backwards orbits respectively.

*Remark* 2.1. Notice that the definition of a flow implies that two distinct orbits $\mathcal{O}(x), \mathcal{O}(y)$ are either disjoint or coincide (exercise).

*Remark* 2.2. Discrete Time Dynamical Systems and Continuous Time Dynamical Systems seen as (semi-)groups of transformations are just special cases of even more general *Group Actions* acting on some space $X$. In these notes we will consider only the cases mentioned above but the theory can be extended to more general groups, e.g. systems with "complex time" parameterized by $\mathbb{C}$.

Unlike discrete time dynamical systems, continuous time systems are generally not defined very explicitly as they are not obtained by iterating a fixed map. There are instead basically two main sources of continuous time systems: suspension flows and ODEs. However there are some cases which we can define explicitly and we start with those.

### 2.1.3 Basic examples

*Example* 1 (Translations on $\mathbb{R}$). Let $X = \mathbb{R}$ and $\alpha \in \mathbb{R}$. For every $t \in \mathbb{R}$ define the map $f^t : X \to X$ by

$$f^t(x) = x + \alpha t. \tag{2.3}$$

Notice that if $\alpha \neq 0$ then $\mathcal{O}(x) = \mathbb{R}$ for any $x \in \mathbb{R}$, but the asymptotic behaviour depends on $a$. Indeed,

- if $a > 0$ then $x(t) \to +\infty$ as $t \to +\infty$ and $x(t) \to -\infty$ as $t \to -\infty$;
- if $a = 0$ then $x(t) \equiv 0$;
- if $a < 0$ then $x(t) \to +\infty$ as $t \to -\infty$ and $x(t) \to -\infty$ as $t \to +\infty$.

*Example* 2 (Translations on $\mathbb{R}^2$). Let $X = \mathbb{R}^2$ and $(\alpha_1, \alpha_2) \in \mathbb{R}^2$. Then for every $t \in \mathbb{R}$ define the map $f^t : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f^t(x, y) = (x_1 + \alpha_1 t, x_2 + \alpha_2 t).$$

Then, if $(\alpha_1, \alpha_2) \neq (0, 0)$, each orbit $\mathcal{O}(x_0, y_0)$ is just a straight line through $(x_0, y_0)$ with slope $\alpha := \alpha_2/\alpha_1$

*Remark* 2.3. When working in higher dimensional spaces it will be useful to use the notation $x = (x_1, x_2, x_3, ...x_n)$ to denote the coordinates of the point $x$. This should not be confused with the notation used previously where we denoted $x_n = f^n(z_0)$. We will not use the two notations in the same settings and so this should hopefully not cause confusion.

*Example* 3 (Translations on tori). Let $X = \mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ and $(\alpha_1, \alpha_2) \in \mathbb{R}^2$. Then for every $t \in \mathbb{R}$ define the map $f^t : \mathbb{T}^2 \to \mathbb{T}^2$ by

$$f^t(x_1, x_2) = (x_1 + \alpha_1 t, x_2 + \alpha_2 t) \mod 1. \tag{2.4}$$

Then, if $(\alpha_1, \alpha_2) \neq (0, 0)$, each orbit $\mathcal{O}(x_0, y_0)$ is winds round the torus.

*Example* 4 (Linear flows on $\mathbb{R}$). Let $X = \mathbb{R}$ and $a \in \mathbb{R}$. Then for each $t \in \mathbb{R}$ define the map $f^t : \mathbb{R} \to \mathbb{R}$ by

$$f^t(x) = x(t) = xe^{at}.$$

Notice that for any $a \in \mathbb{R}$ we have $x(0) = 1$. The range of the function $x$ depends on the value of $a$:

- if $a > 0$ then $x(t) \to +\infty$ as $t \to +\infty$ and $x(t) \to 0$ as $t \to -\infty$;
- if $a = 0$ then $x(t) \equiv 1$;
- if $a < 0$ then $x(t) \to +\infty$ as $t \to -\infty$ and $x(t) \to 0$ as $t \to +\infty$.

In particular the range of $x$ is the whole positive real axis if $a \neq 0$ but is just a single point if $a = 0$.

*Example* 5 (Linear flows on $\mathbb{R}^2$). Let $X = \mathbb{R}^2$ and $(a_1, a_2) \in \mathbb{R}^2$. Then for each $t \in \mathbb{R}$ define the map $f^t : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f^t(x, y_0) = (xe^{a_1 t}, ye^{a_2 t}).$$

## 2.2 Basic constructions

Discrete time and continuous time dynamical systems can be related in a very concrete sense over and above the formal common terminology with which they are described. We give here a few examples.

### 2.2.1 Time-one maps

Let $\{f^t\}_{t\in\mathbb{R}}$ be a flow. Consider the map $f = f^1$ given by $f(x_0) = x_1$ which associates to each point $x_0$ its position after one unit of time under the flow. This is called the *time-one map* associated to the flow. We can now consider the discrete time dynamical systems $\{f^n\}_{n\in\mathbb{Z}}$ given by iterating $f$ both forwards and backwards. It is clear that for each $n \in \mathbb{Z}$ the map $f^n$ exists both in the discrete time dynamical systems and in the flow, in fact the embedding of $\mathbb{Z}$ into $\mathbb{R}$ corresponds to an embedding of $\{f^n\}_{n\in\mathbb{Z}}$ into $\{f^t\}_{t\in\mathbb{R}}$ so that the discrete time dynamical system generated by the time-one map is naturally embedded into the flow. We can of course define, in an exactly analogous way, the time-$t$ map, for any $t$.

*Example* 6. Let $X = \mathbb{R}$ and $f^t(x) = xe^{ax}$ for every $t \in \mathbb{R}$ be a flow. Then for $t = 1$ we have $f(x) = \lambda x$ where $\lambda = e^a$, and for every $n$ we have the discrete time dynamical system $f^n(x) = \lambda^n x = (e^a)^n x = e^{an}x$.

*Remark* 2.4. Notice that we can define the time 1 map for any flow or semiflow. On the other hand it is not true that every discrete time dynamical systems can be embedded into a continuous time dynamical system. Consider for example the map

$$f(x) = \lambda x \quad \text{for some } \lambda < 0.$$

Then

$$x_n = f^n(x) = \lambda^n x$$

and therefore the sign of $x_n$ switches between positive and negative at each iteration. The orbit of $x_0$ can therefore clearly not be embedded into any continuous time dynamical system.

### 2.2.2 Poincaré maps

Another imporan t way to relate discrete time and continuous time dynamical systems is via Poincaré sections and Poincaré maps. Let $\{f^t\}_{t\in\mathbb{R}}$ be a flow on a space $X$ and suppose there exists some subset $\Sigma \subset X$ with the property that for all $x \in \Sigma$ there exists *first return time* function

$$\tau(x) := \min\{t > 0 : f^t(x) \in \Sigma\}$$

such that $f^t(x) \notin \Sigma$ for all $t \in (0, \tau(x))$ and $f^{\tau(x)}(x) \in \Sigma$. Then we call $\Sigma$ a Poincaré section for the flow and we define the Poincaré first return time map $F : \Sigma \to \Sigma$ by

$$F(x) = f^{\tau(x)}(x).$$

In some cases one can recover properties of the flow by studying the discrete time dynamical systems generated by the map $F$ on $\Sigma$. Notice for example that any periodic point for $F$ defines a periodic orbit for the flow.

*Example* 7. Recall the translation flow on the torus defined in (2.4) above. Supposing without loss of generality that $\alpha_1 \neq 0$ it is easy to see that the set $\Sigma = \{0\} \times \mathbb{S}^1 \subset \mathbb{T}^2$ is a Poincaré section and every point returns ti $\Sigma$ in this case with a constant return time $\tau(x) = 1/\alpha_1$. Moreover, it is not difficult to see that $F$ is nothing else than a *circle rotation* by an angle $\alpha = \alpha_1/\alpha_2$ and therefore our understanding of circle rotations can help us to understand the dynamics of translation flows on the torus. In particular, if $\alpha$ is rational then every point is periodic for the flow whereas if $\alpha$ is irrational then every orbit is dense in $\mathbb{T}^2$ for the flow, see Exercise **??**. Indeed, this was one of the motivations for Poincaré's interest in circle rotations.

*Remark* 2.5. While every flow and semi-flow has a corresponding time-1 map, it is not at all the case that every flow or semi-flow has a Poincaré map. A trivial example are flows with no recurrence, such as the linear translation flows on $\mathbb{R}$ defined in (2.3) which clearly cannot admit Poincaré map. In many cases it is a relevant and non-trivial problem whether given flows admit Poincaré sections and Poincaré maps.

### 2.2.3 Suspension flows

We can also go in the other direction, and build a flow out of a discrete time dynamical system defined on some set $X$. Indeed, let $f : X \to X$ be a map and define an *extended phase space*

$$\widehat{X} := X \times [0, 1]/\sim$$

where the relation $\sim$ identifies the point $(x, 1)$ with the point $(f(x), 0)$. We can then define a flow on $\widehat{X}$ by simple vertical translation with constant speed 1. More precisely, for any point $(x_0, y_0) \in \widehat{X}$ if $t > 0$ we define $f^t(x_0, y_0)$ by first translating the point "upwards" by time $1 - y_0$ until it hits the "roof", then identifying this point with the corresponding point $f(x_0)$ on the "base", then translating again upwards and continuing in this way until we reach time $t$. Notice that the original map $f$ is exactly the time-one map of this flow which in this special case also coinmcides with the first return Poincaré map on the cross section $X$.

*Example* 8. Let $X = \mathbb{S}^1$ and $f(x) = x + \alpha \mod 1$ be a circle rotation. Then this construction yields what is essentially a torus with a translation flow for which $f$ is the first return Poincaré map.

A further generalization of the constant-time suspension flows above can be constructed by considering once again a map $f : X \to X$ and a "roof" or "return-time" function $r : X \to \mathbb{R}^+$ and then defining the extended phase space

$$\widehat{X}_r := \{(x,t) : x \in X, t \in [0, r(x)]\}/\sim \tag{2.5}$$

where this time the relation $\sim$ identifies the point $(x, r(x))$ with the point $f(x), 0)$. The flow can be defined then in a very similar way to that of the constant-time suspension flow except that the time it takes to return to the base is variable and depends on the base point. It is a very general construction and one of the easiest ways of construting semi-flows.

*Example* 9. Let $X = \mathbb{S}^1$, $f(x) = 10x \mod 1$, and $r : \mathbb{S}^1 \to \mathbb{R}^+$ an arbitrary continuous function. Then we can use the construction described above to construct a semi-flow which has $f$ as a Pincaré first return map.

# Chapter 3

# ODEs

We now come to the most classical and arguably most important source of flows. To motivate the somewhat abstract definition we first make some general observations concerning flows.

### 3.0.1 Vector fields and integral curves

Let $\{f^t\}_{t\in\mathbb{R}}$ be a flow on $\mathbb{R}^n$ and suppose that the family of maps $f^t$ depends

1. *differentiably* on the parameter $t$ in the sense that the map $x : t \mapsto f^t(x)$. is differentiable with respect to $t$ for every $t \in \mathbb{R}$, and
2. *continuously* on the point $x$ in the sense that for each fixed $t$ the map $f^t : \mathbb{R} \to \mathbb{R}$ is a continuous.

The differentiability with respect to $t$ means in particular that the orbit of each point is a differentiable curve and that for each $t$ the derivative

$$\dot{x}(t) = (\dot{x}_1(t), ..., \dot{x}_n(t)).$$

is a well defined vector in $\mathbb{R}^n$ tangent to the curve of the orbit in $x(t)$. Moreover, recall Remark 2.1, each point of $\mathbb{R}^n$ belongs to one and only one orbit and therefore the flow defines a vector $V(x)$ at each point of $\mathbb{R}^n$. We make the following definition. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be an open set.

**Definition 3.1.** A *vector field* on $\mathcal{U}$ is a function $V : \mathcal{U} \to \mathbb{R}^n$.

The discussion above says that every flow on $\mathbb{R}^n$ defines a vector field. A natural but very deep and non-trivial question is the converse question, for which we have the following answer.

**Theorem 3.2.** *Every Lipschitz vector field on $\mathbb{R}^n$ defines a flow.*

Recall that a continuous map $V : \mathbb{R}^n \to \mathbb{R}^n$ is *locally Lipschitz* at $c \in \mathbb{R}^n$ if there exists a neighbourhood $U = U(c)$ and a constant $\kappa = \kappa(c) > 0$ such that for every $x, y \in U$ we have $\|V(x) - V(y)\| \leq \kappa\|x - y\|$. $V$ is *Lipschitz* if there exists $\kappa > 0$ such that for all $x, y \in \mathbb{R}^n$ we have $\|V(x) - V(y)\| \leq \kappa\|y - z\|$ (i.e. if it is locally Lipschitz at every point $c \in \mathbb{R}^n$ with a uniform constant $\kappa$).

*Example* 10. Easy examples of Lipschitz functions are functions whose derivative is uniformly bounded. The function $V(x) = x^2$ is locally Lipschitz but not Lipschitz since whose derivative is unbounded at infinity and $V(x) = \sqrt{x}$ is not locally Lipschitz at 0 where the derivative is unbounded.

To formulate this question precisely we need to introduce some definitions. Let $I \subseteq \mathbb{R}$ be an open interval. If $x : I \to \mathcal{U}$ is a $C^1$ function, for $t \in I$ we write $x(t) = (x_1(t), ..., x_n(t))$ where $x_i : I \to \mathbb{R}$ are the coordinate functions of $x$. The derivative $\dot{x}(t)$ of $x$ with respect to $t$ is then the vector $\dot{x}(t) = (\dot{x}_1(t), ..., \dot{x}_n(t))$ where $\dot{x}_i(t)$ are the derivatives of the coordinate functions $x_i$ wrt $t$.

**Definition 3.3.** A function $x : I \to \mathcal{U}$ is an *integral curve* of a vector field $V$ if

$$\dot{x}(t) = V(x(t)) \tag{3.1}$$

for every $t \in I$. Equation (3.1) is sometimes written in the form $\dot{x} = V(x)$ and referred to as an (autonomous) *ordinary differential equation.* A function $x$ satisfying (3.1) is called a *(local) solution* of the differential equation. A function $x$ satisfying (3.1) with $I = \mathbb{R}$ is called a global solution of the differential equation.

*Remark* 3.4. The definition of an integral curve implies not only that it is geometrically tangent to the vector field $V$ at every point, but that it's parametrization is such that its velocity at every point is exactly the vector given by the vector field at that point.

*Remark* 3.5. A *non-autonomous* ordinary differential equation is an equation of the form $\dot{x} = V(x, t)$ where $V : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a one-parameter family of vector fields. There are many similarities but also many differences between autonomous and non-autonomous differential equations but we will not explore them here.

This prompts the following set of questions, known as the *Cauchy problem*: given a vector field $V : \mathcal{U} \to \mathbb{R}^n$ and a point $x_0 \in \mathcal{U}$ we write

$$\begin{cases} \dot{x} = V(x) \\ x(0) = x_0. \end{cases} \tag{3.2}$$

We can then ask: *i)* does there exist an interval $I \subseteq \mathbb{R}$ and a $C^1$ function $x : I \to \mathcal{U}$ which is a solution of (3.13), i.e. such that $\dot{x} = V(x)$ and $x(0) = x_0$? *ii)* is this solution unique? *iii)* what is the maximal interval $I$ on which the solution can be defined?

*Remark* 3.6. Notice that if $x : I \to \mathcal{U}$ is a solution of (3.13) on some interval $I$ containing 0, then the restriction of $x$ to any subinterval $J \subset I$ also containing 0, is also a solution. We clearly do not consider these distinct solutions and thus say that a solution $x : I \to \mathcal{U}$ is *unique* if any other solution $y : J \to \mathcal{U}$ coincides with $x$ on the intersection of their respective domains of definition $I \cap J$.

In Section 3.0.2 we give some easy and explicit examples for which this question can be answered positively. In Section 3.0.3 we give some basic counterexamples which show that in somecases uniqueness and global existence fails. Finally in Section 3.1 we give classical conditions for existence and uniqueness of solutions and for the existence of a flow.

### 3.0.2 Basic examples

*Example* 11. Let $a \in \mathbb{R}$ and let $V : \mathbb{R} \to \mathbb{R}$ be the constant vector field $V(x) = a$. Consider the differential equation $\dot{x} = V(x)$ or, equivalently,

$$\dot{x} = a. \tag{3.3}$$

Then it is easy to see that, for any $c \in \mathbb{R}$, the function $x : \mathbb{R} \to \mathbb{R}$ given by

$$x(t) = c + at \tag{3.4}$$

satisfies $\dot{x}(t) = a$ and is therefore a solution.

*Remark* 3.7. Notice that for many choices of $a$ and $c$ the *solution curve*, i.e. the image $x(\mathbb{R})$, is the same. This however does not make them the same solution as they are distinct functions, i.e. distinct solutions, with the same image.

*Example* 12. Let $a \in \mathbb{R}$ and let $V : \mathbb{R} \to \mathbb{R}$ be the vector field $V(x) = ax$. Consider the differential equation $\dot{x} = V(x)$ or, equivalently,

$$\dot{x} = ax. \tag{3.5}$$

Then it is easy to see that for any $c \in \mathbb{R}$ the function

$$x(t) = ce^{at} \tag{3.6}$$

is an integral curve of $V$ and therefore a solution to the differential equation (3.5). Indeed, differentiating $x$ with respect to $t$ we get

$$\dot{x}(t) = ace^{at} = ax(t)$$

We have that $x(0) = c$ is the *initial condition* of the solution, i.e. the position of the solution at time $t = 0$, and a very similar analysis as above can be carried out to study the range of $x$ and its behaviour as $t \to \pm\infty$. Notice that the three specific cases above are just special cases of (3.6) with initial conditions $c = 1$, $c = -1$ and $c = 0$ respectively.

*Example* 13. Let $a = (a_1, a_2) \in \mathbb{R}^2$ and let $V : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field $V(x) = a$ or equivalently $V(x_1, x_2) = (a_1, a_2)$ where $x = (x_1, x_2)$. Consider the differential equation $\dot{x} = V(x)$ we can write $(\dot{x}_1, \dot{x}_2) = (a, b)$ or even

$$\begin{cases} \dot{x_1} = a_1 \\ \dot{x_2} = a_2 \end{cases} \tag{3.7}$$

This corresponds to choosing a constant vector $a = (a_1, a_2)$ at every point. It is then easy to see that for any initial condition $c = (c_1, c_2) \in \mathbb{R}^2$ function $x : \mathbb{R} \to \mathbb{R}^2$ given by

$$x(t) = (x_1(t), x_2(t)) = (c_1, c_2) + (a_1 t, a_2 t) = (c_1 + a_1 t, c_2 + a_2 t) \tag{3.8}$$

is a solution of (3.7) since $\dot{x}_1 = a_1$ and $\dot{x}_2 = a_2$. For $a = (a_1, 0)$ the solution curves are just horizontal lines, for $a = (0, a_2)$ they are vertical lines, and in the general case $a = (a_1, a_2)$ with $a_1, a_2 \neq 0$ they are straight lines with slope $a_2/a_1$.

*Example* 14. Let $a = (a_1, a_2) \in \mathbb{R}^2$ and let $V : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field $V(x) = ax$ or equivalently $V(x_1, x_2) = (a_1 x_1, a_2 x_2)$ where $x = (x_1, x_2)$. Consider the differential equation $\dot{x} = V(x)$ or $\dot{x} = ax$, which is the exact analogue of (3.5) with the scalars $a, x$ replaced by vectors, and which, for clarity, we can write

$$\begin{cases} \dot{x_1} = a_1 x_1 \\ \dot{x_2} = a_2 x_2 \end{cases} \tag{3.9}$$

This corresponds to choosing a constant vector $ax = (a_1 x_1, a_2 x_2)$ at every point. It is a useful exercise to try this for various values of $a = (a_1, a_2)$. It is then easy to see that for any initial condition $c = (c_1, c_2) \in \mathbb{R}^2$ function $x : \mathbb{R} \to \mathbb{R}^2$ given by

$$x(t) = (x_1(t), x_2(t)) = (c_1 e^{a_1 t}, c_2 e^{a_2 t}) \tag{3.10}$$

is a solution of (3.9), this can be checked just as in the one-dimensional case. Notice however that the geometry of the solution curves depends very much on the signs of $a_1, a_2$.

### 3.0.3 Basic counterexamples

*Example* 15 (Non-uniqueness). Consider the differential equation

$$\dot{x} = 3x^{2/3} \tag{3.11}$$

The two functions

$$x(t) \equiv 0 \quad \text{and} \quad y(t) = t^3$$

satisfy $x(0) = y(0) = 0$ and are both solutions to (3.11). Indeed, since $x$ is constant, its derivative is zero and therefore for any $t \in \mathbb{R}$ we have $\dot{x}(t) = 0 = V(0) = V(x(t))$. On the other hand we also have $\dot{y}(t) = 3t^2 = 3(t^3)^{2/3} = 3(y(t))^{2/3} = V(y(t))$. Thus both $x$ and $y$ solve the Cauchy problem and we do not have uniqueness of solutions. Notice that this does not contradict the fundamental theorem of ODEs. The vector field here is given by the function $V(x) = 3x^{2/3}$ which is continuous and differentiable outside the origin with $V'(x) = 2x^{-1/3}$. Notice that $V'(x) \to \infty$ as $x \to 0$ so that $V$ is not locally Lipschitz at 0. Notice that in this case the two solutions are geometrically distinct. One is just a fixed point at the origin, whereas the other maps to the entire real line.

*Example* 16 (Non-global solutions)*.* Consider the differential equation

$$\dot{x} = 1 + x^2 \tag{3.12}$$

For any initial condition $x_0$ the function

$$x(t) = \tan(t + c) \qquad \text{where } c = \tan^{-1}(x_0)$$

is solution of (3.12). Indeed, $x(0) = \tan(c) = \tan(\tan^{-1}(x_0)) = x_0$ and $\dot{x} = \dot{x}(t) = 1 + \tan^2(t+c) = 1 + (\tan(t+c))^2 = 1 + (x(t))^2 = 1 + x^2$. Notice however that these solutions are not globally defined. Indeed, the function tan is only well defined in the interval $(-\pi/2, \pi/2)$ and so the solution $x$ is defined only on the interval $I = (-c - \pi/2, -c + \pi/2)$. Notice however that geometrically the solution maps to the whole real line since $x(t) \to \pm\infty$ as $t \to -c \pm \pi/2$. Thus the trajectory *goes to infinity in finite time.*

## 3.1 Existence and uniqueness of solutions

The following is the so-called *fundamental theorem of ordinary differential equations.* To simplify the statement we consider the case where $\mathcal{U} = \mathbb{R}^n$ so that the vector field is defined on all of $\mathbb{R}^n$.

**Theorem 3.8.** *Let $V : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field. For every $x_0 \in \mathbb{R}^n$:*
*i) there exists a local solution $x : I \to \mathbb{R}^n$ for the Cauchy problem* (3.13)*;*
*ii) if $V$ is locally Lipschitz then this solution is unique;*
*iii) if $V$ is Lipschitz this solution is a global solution.*
*In case iii) the solutions depends continuously on the initial condition.*

*Remark* 3.9. If $V$ is Lipschitz and $x : \mathbb{R} \to \mathbb{R}^n$ and $y : \mathbb{R} \to \mathbb{R}^n$ are two global solutions with initial conditions $c, d \in \mathbb{R}^n$ respectively, then continuous dependence means that for any given $t \in \mathbb{R}$ and any $\epsilon > 0$ there exists $\delta = \delta(\epsilon, t) > 0$ such that $|c - d| < \delta$ guarantees $|x(t) - y(t)| < \epsilon$. We emphasize that $\delta$ depends on $t$ as well as $\epsilon$; in general two global solutions with arbitrarily close initial conditions may eventually diverge, but this does not contradict the continuous dependence of solutions.

## 3.2 Exercises

### 3.2.1 Vector Fields

**Exercise 3.2.1.** For $a, b \in \mathbb{R}$ consider the vector field $V(x_1, x_2) = (ax_1, bx_2)$ on $\mathbb{R}^2$.

(a) Sketch the vector field for the following two cases:
  (i) $a = -1$, $b = 3$
  (ii) $a = -1$, $b = -2$

(b) Add to the sketches the flow lines which pass through the point $(1, 1)$.

(c) Verify that $x(t) = (e^{at}, e^{bt})$ is a solution to the Cauchy problem

$$\begin{cases} \dot{x} = V(x) \\ x(0) = (1, 1) \end{cases}$$

(d) Is this solution unique? If so, why? If not, can you find another solution?

**Exercise 3.2.2.** Consider the vector field $V(x_1, x_2) = (1, \sqrt{|x_2|})$ on $\mathbb{R}^2$.

(a) Sketch the vector field.

(b) Add to the sketch the flow line which passes through $(0, 1)$.

(c) Verify that[1] $x(t) = (t, \text{sgn}(t)\frac{1}{4}t^2)$ is a solution to the Cauchy problem $\dot{x} = V(x)$ with $x(0) = (0, 1)$.

(d) Is this solution unique? If so, why? If not, can you find another solution?

**Exercise 3.2.3.** Consider the vector field $V(x_1, x_2) = (2, x_2^2)$ on $\mathbb{R}^2$.

(a) Sketch the vector field.

(b) Add to the sketch the flow line which passes through $(0, 1)$.

(c) Verify that $x(t) = (2t, \frac{1}{1-t})$ is a solution to the Cauchy problem $\dot{x} = V(x)$ with $x(0) = (0, 1)$.

(d) Is the solution in (c) unique? If so, why? If not, can you find another solution? Is the solution valid for all $t \in \mathbb{R}$?

### 3.2.2 Existence and Uniqueness

**Theorem 3.10** (Existence and Uniqueness)**.** *Let $y_0 \in \mathbb{R}^n$, $b > 0$. Suppose that $V$ is a Lipschitz vector field on $D = [y_0 - b, y_0 + b]^n \subset \mathbb{R}^n$. Then there exists a*

[1]The function $\text{sgn}(t)$ is equal to 1 if $t \geq 0$ and equal to $-1$ otherwise.

*unique solution to the Cauchy problem*

$$\begin{cases} \dot{y}(t) = V(y(t)) \\ y(0) = y_0 \end{cases} \tag{3.13}$$

*on the interval* $[-a, a]$, *for some* $a > 0$.

In a series of exercises we will prove Theorem 3.10. Since $V$ is Lipschitz there exists $L > 0$ such that $|V(x) - V(y)| \leq L|x - y|$ for all $x, y \in D$. Let $M = \sup\{|V(x)| : x \in D\}$. Choose

$$a < \min\left\{\frac{1}{L}, \frac{b}{M}\right\}.$$

Denote by $\mathcal{Y}$ the space of all functions $y : [-a, a] \to D$ which are continuous. For $y_1, y_2 \in \mathcal{Y}$, define

$$d(y_1, y_2) = \max_{t \in [-a,a]} |y_1(t) - y_2(t)|.$$

**Exercise 3.2.4.** Show that $(\mathcal{Y}, d)$ is a complete metric space using the following steps.

1. Show that $d$, as defined above, is a metric.
2. Show that the limit of any Cauchy sequence is a function which takes values in $D$.
3. Observe that any Cauchy sequence converges uniformly. Show that the uniform limit of continuous functions is continuous using the following steps.
   (a) Suppose that the sequence of continuous $\{y_n\}$ converges uniformly to $y$. Then for $\epsilon > 0$, there exists an $N$ such that for $n \geq N$, $|y_n - y| \leq \frac{\epsilon}{3}$.
   (b) Since each $y_n$ is continuous, then there exists a $\delta$ such that for $|t_1 - t_2| < \delta$, $|y_n(t_1) - y_n(t_2)| < \frac{\epsilon}{3}$.
   (c) Consider $|y(t_1) - y(t_2)|$. By adding and subtracting both $y_n(t_1)$ and $y_n(t_2)$ and using the triangle inequality, show that $y$ is continuous.

We define the operator $T : \mathcal{Y} \to \mathcal{Y}$,

$$(Ty)(t) = y_0 + \int_0^t V(y(s))\, ds.$$

**Exercise 3.2.5.** Show that $T$ is well-defined. Hint: Is $t \mapsto (Ty)(t)$ continuous? Is $(Ty)(t) \in D$ for all $t \in [-a, a]$?

**Exercise 3.2.6.**

1. Show that if $y$ is a solution to (3.13) then $y$ is a fixed point of $T$.
2. Show that if $y$ is a fixed point of $T$, then $y$ is a solution to (3.13).

**Exercise 3.2.7.**

1. Show that there exists $c \in (0,1)$ such that $d(Ty_1, Ty_2) \leq c\, d(y_1, y_2)$ for all $y_1, y_2 \in \mathcal{Y}$.
2. Use the Contraction Mapping Theorem to conclude the proof of the existence and uniqueness of a local solution.

# Chapter 4

# Conjugacy and Structural Stability

We are now ready to begin to formulate the basic problem of Dynamical Systems, that of the *classification* and of *structural stability* of dynamical systems.

## 4.1 Classification of Dynamical Systems

We start by introducing some fundamental notions used to describe a dynamical system and then discuss when two systems can be considered equivalent and formulate the problem of the stability. For simplicity we will mainly focus on discrete time dynamical systems, but most ideas can easily be translated to the continuous time setting.

### 4.1.1 Conjugacy

The first and most fundamental concept in any problem of classification is to formulate a notion of equivalence. As we shall see there are various notions which can be useful and which define equivalence with various degrees of strength. The most basic form is the following.

Let $X, Y$ be two sets, $f : X \to X$ and $g : Y \to Y$ two maps.

**Definition 4.1.** $f, g$ are *conjugate* if there exists a *bijection* $h : X \to Y$ such that

$$h \circ f = g \circ h.$$

The conjugacy condition says that $h$ maps orbits to orbits in a consistent way. Indeed, if $f, g$ are conjugate, we have $f = h^{-1} \circ g \circ h$ and therefore for any $n$ for which $f^n$ and $g^n$ are both defined, we have

$$f^n = (h^{-1} \circ g \circ h)^n = h^{-1} \circ g \circ h \circ \cdots \circ h^{-1} \circ g \circ h = h^{-1} \circ g^n \circ h.$$

Thus all iterates of $f, g$ are also conjugate. Thus the conjugacy is really a relationship between the dynamical systems induced by $f, g$ respectively. Is it easy to check that conjugacy defines an equivalence relation in the space of all discrete time dynamical systems (Exercise 4.3.1, Item 1) and thus is an acceptable notion of equivalence. It is however a very weak notion: it guarantees that two conjugate systems have corresponding sets of periodic points (Exercise 4.3.2, Item 1) but in fact not much else. As we shall see from examples below it is possible to conjugate systems which we do not really want to consider equivalent. We thus need a stronger form of conjugacy and this requires also some additional structure on the maps $f, g$.

### 4.1.2 Topological conjugacy

We now assume that $X, Y$ are topological spaces. This will allow us to formulate a much stronger and more meaningful notion of equivalence, by imposing an extra condition on the conjugacy condition introduced above. Let $f : X \to X$ and $g : Y \to Y$ be two maps. In the context of maps on topological spaces it is natural to consider continuous maps, and many of the maps we will consider will be continuous, but this is not strictly required by the definition.

**Definition 4.2.** $f, g$ are *topologically conjugate* if they are conjugate and the conjugacy $h$ is a *homeomorphism*.

It is again easy to see that this is an equivalence relation (Exercise 4.3.1, Item 2). The crucial difference is that topological conjugacies also preserve limit sets (Exercise 4.3.2, Item 2), in the sense that if $f, g$ are topologically conjugate, for every $x \in X$ we have

$$h(\omega(x)) = \omega(h(x)).$$

As we shall see, this is a much stronger form of conjugacy than simple conjugacy. If two maps are topologically conjugate then they are conjugate but the converse is false. Thus topological conjugacy classes are a *refinement* of standard conjugacy classes.

### 4.1.3 Differentiable conjugacy

If the spaces $X, Y$ are differentiable manifolds such as $\mathbb{R}^n$ we can take the notion of conjugacy even further. Suppose $f : X \to X$ and $g : Y \to Y$ are two maps $C^1$ maps.

**Definition 4.3.** $f, g$ are $C^1$ *conjugate* if they are conjugate and the conjugacy $h$ is a $C^1$ *diffeomorphism*.

Also in this case it is easy to check that $C^1$ conjugacy is an equivalence relation. Any $C^1$ diffeomorphism is of course also in particular a homeomorphism and so $C^1$ conjugacies preserve limit sets, but they also preserve even more structure, in particular the value of the derivative at fixed points (Exercise 4.3.2, Item 3)

*Remark* 4.4. There are advantages and disadvantages of using weaker or strong forms of conjugacies and the correct level may depend on the specific question. Weaker forms mean that it is easier for two systems to be equivalent but that may include cases which really we feel should be distinct, on the other hand stronger forms may distinguish too much including between systems which for certain purposes In each case more and more structure is preserved. The correct notion of conjugacy will depend on the specific setting and the questions of interest. In general however differentiable conjugacies preserve so much structure that even systems which seems very similar may not be conjugate, and in many situations topological conjugacies seem like the appropriate compromise between preserving a sufficient amount of structure and sufficiently large equivalence classes.

## 4.2 Structural stability and bifurcations

The notions of conjugacy or equivalence between two dynamical systems are particularly significant in conjunction with the concept of a *perturbation*, i.e. a "small" change in the system. What happens if we change the system a very little bit? *Is the perturbed system conjugate to the original system?* The answer to this question depends on the kind of conjugacy we require but also on the kind of perturbation we allow, some perturbations can create more damage than others. This is formalized by a choice of metric or topology on the space of dynamical systems. In the coming chapters we will consider several examples and give precise definitions of the kinds of perturbations and the conjugacies appropriate to various settings. At the moment we give a somewhat "conceptual" definition.

**Definition 4.5.** A dynamical system is *structurally stable* (with respect to a given notion of conjugacy and with respect to a given topology on the appropriate space of dynamical systems) if it lies in the *interior* (with respect to the topology) of its equivalence class (with respect to the given notion of conjugacy).

Thus, structural stability means that a sufficiently small perturbation (in the chosen topology) does not modify the features of the system (in the chosen level of conjugacy). If the system is not structurally stable then there exist other dynamical systems "arbitrarily close" to the original one which are not conjugate and thus "different". In this case we say that the system is undergoing a *bifurcation*.

*Example* 17. Consider the map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 + 1/4$. It is easy to check that $f$ has exactly one fixed point $p = 1/2$ and that the graph of $f$ is tangent to the diagonal at the point $(1/2, 1/2)$. Then it is clear that there any arbitrarily small perturbations can push the graph of $f$ fully above the diagonal, thus destroying the fixed point or, conversely, push it a little bit downwards so that it intersects the diagonal twice and so has 2 fixed points. In both cases the number of fixed points has changed and so the new system cannot be conjugate the the original one. We will see form the arguments below that the perturbed systems on the other hand are structurally stable.

## 4.3 Exercises

**Exercise 4.3.1.** Prove the following statements:

1. Conjugacy is an equivalence relation.
2. Topological conjugacy is an equivalence relation
3. $C^1$ conjugacy is an equivalence relation.

**Exercise 4.3.2.** Show that

1. Conjugacy preserves periodic points, i.e. if $f, g$ are conjugate, the conjugacy maps periodic points to periodic points.
2. Topological conjugacy preserves limits sets, i.e. if $f, g$ are topologically conjugate, for every $x \in X$ we have $h(\omega(x)) = \omega(h(x))$.
3. $C^1$ conjugacy preserves the derivative at fixed points, i.e. if $f(p) = p, g(q) = q$ and $h(p) = q$ then $f'(p) = q'(q)$.

# Part II

# One-dimensional diffeomorphisms

# Chapter 5

# One-Dimensional Linear Maps

As a first example of applications of the concepts introduced above we consider one-dimensional linear maps. The situation in this setting is particularly simple but precisely for that reason it is a good class of maps through which to highlight some fundamental ideas and techniques.

## 5.1 Dynamics of one-dimensional linear maps

The only linear maps in one-dimension are the scalar maps $A : \mathbb{R} \to \mathbb{R}$ given by

$$A(x) = ax$$

for some $a \in \mathbb{R}$. Iterates of $A$ clearly have the form

$$A^n(x) = a^n x$$

and this allows us to easily and systematically study the dynamics and the alpha and omega limits for various values of the parameter $a$.

**Definition 5.1.** A one dimensional linear map $A(x) = ax$ is

1. *invertible* if $a \neq 0$
2. *hyperbolic* if $a \neq \pm 1$;
3. *orientation preserving* if $a > 0$;
4. *orientation reversing* if $a < 0$;
5. *contracting* if $|a| \in (0, 1)$
6. *expanding* if $|a| > 1$

The dynamical features of the map (e.g. fixed and periodic points, limit sets) depend essentially on which of the above classes it belongs to (Exercise 5.4.1).

## 5.2 Conjugacy classes

The main purpose of this chapter is to prove the following

**Theorem 5.2.** *Let $A(x) = ax, B(x) = bx$ with $a, b \neq 0$ and $a \neq b$. $A$ and $B$ are:*

1. ***conjugate*** *if and only if they are both hyperbolic and have the same orientation;*
2. ***topologically conjugate*** *if and only if they are conjugate and are either both contracting or both expanding;*
3. *not* $\mathbf{C^1}$ ***conjugate****.*

*Remark* 5.3. Notice that the case $a = 0$ is a degenerate case where all points map to the original after one iterate. Clearly in this case $A$ cannot be conjugate to any other linear map $B$ with $b \neq 0$. Also, every map is always automatically conjugate to itself via the identity map, so we exclude this case from the above.

*Remark* 5.4. This result illustrates particularly clearly the difference between the three levels of conjugacy. In particular it shows that simple conjugacy is quite a weak notion of equivalence, since for example $A(x) = 2x$ and $B(x) = x/2$ are conjugate even though one is expanding and all orbits go to infinity in forward time whereas the other ic onctracting and all orbits go to 0 in forward time. On the other hand $C^1$ conjugacy is extremely strong and no two distinct linear maps are conjugate in this way. Topological conjugacy is on the whole a reasonable intermediate notion which conjugates maps which it seems reasonable to consider equivalent, and distinguishes maps which it seems reasonable to consider distinct. As we shall see below, this will also apply to more general situations, and topological conjugacy turns out to be the most convenient notion of equivalence to use in most situations.

To prove Theorem 5.2 we shall use a powerful method for the construction of conjugacies, called the method of *fundamental domains.* We explain it first in an abstract setting.

**Definition 5.5.** Let $X$ be a set and $f : X \to X$ be an invertible map. A subset $X' \subseteq X$ is *invariant* if $f(X') = X'$.

*Example* 18. Any fixed point is clearly invariant, as is the orbit of a periodic point. In fact any full orbit $\theta(x_0) = \{f^n(x_0)\}_{n\in\mathbb{Z}}$ is invariant and therefore also any union of orbits. In some cases there are entire regions that are invariant in a natural way, for example for the linear map $A(x) = 2x$ both strictly positive and strictly negative semi-axes are invariant.

If $X'$ is an invariant subset then we can consider the dynamics of the map $f$ restricted to $X'$, since any point in $X'$ maps to $X'$ both in forward and backward iterates. We denote by $f|_{X'}$ the restriction of $f$ to $X'$.

**Definition 5.6.** A subset $\mathcal{U} \subseteq X'$ is a *fundamental domain* for $f|_{X'}$ if for every $x \in X'$ there exists a *unique* time $\tau = \tau(x) \in \mathbb{Z}$ such that $f^\tau(x) \in \mathcal{U}$.

*Example* 19. If $X' = \theta(x_0)$ is the full orbit of a single non-periodic point, then any point in this orbit is a fundamental domain for $X'$ (notice that if $x_0$ is fixed or periodic then $X' = \theta(x_0)$ is invariant but has no fundamental domains since each point returns to every other point infinitely many times). If $A(x) = ax$ is a linear map with $a \in (0, 1)$, then for any $x_0 > 0$ the half-open, half closed interval $(ax_0, x_0]$ is a fundamental domain for the positive semi-axis (Exercise 5.4.2).

It turns out that the problem of establishing a conjugacy between two systems can be essentially reduced to that of finding fundamental domains. Suppose $f : X \to X$ and $g : Y \to Y$ are two invertible maps, $X' \subseteq X, \mathcal{Y}' \subseteq Y$ invariant sets, and $\mathcal{U}, \mathcal{V}$ fundamental domains for $f|_{X'}, g|_{Y'}$ respectively.

**Lemma 5.7.** *Let $\tilde{h} : \mathcal{U} \to \mathcal{V}$ be a bijection, then $f|_{X'}$ and $g|_{Y'}$ are conjugate.*

*Proof.* Exercise 5.4.4 □

*Proof of Theorem 5.2.* From Lemma 5.7 we get the first item in Theorem 5.2 (Exercise 5.4.5). To prove the second item we need to show that if $\tilde{h}$ can be chosen to be a homeomorphism then the conclusions of Lemma 5.7 can be strenghtened to give a topological conjugacy. This is not so easy to show in the full generailty of abstract topological spaces, but can be shown in the setting of linear maps which is what we need here, giving the second item in Theorem 5.2 (Exercise 5.4.7). Finally, for the third item, notice that the origin is a fixed point for every linear map $A(x) = ax$ and the derivative of $A$ at every point (and thus in particular at the fixed point) is $A'(x) \equiv a$. By Exercise 4.3.2 a $C^1$ conjugacy preserves the derivative at fixed points and therefore any two distinct linear maps cannot be $C^1$ conjugate. □

## 5.3 Structural stability

Theorem 5.2 also helps to illustrate the notion of structural stability in this very simple setting. Letting $\mathcal{L}(\mathbb{R}^1)$ denote the space all one-dimensional linear maps. For two maps $A(x) = ax$ and $B(x) = bx$ we define a metric $d(A, B) := |a - b|$.

**Theorem 5.8.** *Let $A \in \mathcal{L}(\mathbb{R}^1)$. Then*

1. *$A$ is not structurally stable with respect to $C^1$ conjugacy;*
2. *$A$ is structurally stable with respect to conjugacy (and topological conjugacy) if and only if it is invertible and hyperbolic.*

*Proof.* Exercise 5.4.8 □

## 5.4 Exercises

**Exercise 5.4.1.** Let $A : \mathbb{R} \to \mathbb{R}$ be a linear map $A(x) = ax$ for some $a \in \mathbb{R}$. Describe the dynamics (fixed points, periodic points, omega and alpha limits sets) in the following cases: *i)* $a = 0$; *ii)* $a = \pm 1$; *iii)* $|a| < 1$; *iv)* $|a| > 1$. In cases *iii)* and *iv)* explain the difference in the dynamics in the orientation preserving and orientation reversing cases.

**Exercise 5.4.2.** Let $A(x) = ax$ be a linear map with $a \in (0, 1)$.

1. Show that each of the positive and the negative semi-axes are invariant;
2. Show that for any $x_0 > 0$ the interval $(ax_0, x_0]$ is a fundamental domain for the positive semi-axis; [*Hint:* notice that the images $A^n(ax_0, x_0]$ are pairwise disjoint and their union is the positive semi-axis]
3. Show that for any $y_0 < 0$ the interval $[y_0, ay_0)$ is a fundamental domain for the negative semi-axis;
4. Show that $(ax_0, x_0] \cup [y_0, ay_0)$ is a fundamental domain for $\mathbb{R} \setminus \{0\}$;
5. Find analogous fundamental domains for $a > 1$.

**Exercise 5.4.3.** Let $A(x) = ax$ be a linear map with $a \in (-1, 0)$.

1. Show that each of the positive and the negative semi-axis is *not* an invariant set, but their union $\mathbb{R} \setminus \{0\}$ *is* an invariant set;
2. Find a fundamental domain for $\mathbb{R} \setminus \{0\}$;
3. Answer both items above in the case $a < -1$.

**Exercise 5.4.4.** Suppose $f : X \to X, g : Y \to Y$ are invertible maps, $X' \subseteq X, \mathcal{Y}' \subseteq Y$ invariant sets, and $\mathcal{U} \subset X', \mathcal{V} \subset Y'$ fundamental domains for $f|_{X'}, g|_{Y'}$ respectively. For every $x \in X'$, let $\tau(x)$ denote the unique time for which $f^{\tau}(x) \in \mathcal{U}$. Let $\tilde{h} : \mathcal{U} \to \mathcal{V}$ be a bijection and for every $x \in X'$ let

$$h(x) := g^{-\tau(x)} \circ \tilde{h} \circ f^{\tau(x)}(x). \tag{5.1}$$

1. Show that $h : X' \to Y'$ is a bijection, i.e. that it is injective and surjective;
2. Show that $h$ is a conjugacy between $f|_{X'}, g|_{Y'}$; [*Hint:* Compute $h \circ f(x) = h(f(x))$ using the formula above and the observation that $\tau(f(x)) = \tau(x) - 1$]

**Exercise 5.4.5.** Let $A(x) = ax, B(x) = bx$ with $a, b \neq 0$ and $a \neq b$.

1. Suppose $A, B$ are conjugate.
   (a) Show that $a, b \neq \pm 1$ and therefore $A, B$ are both hyperbolic.
   (b) Show that $A, B$ have the same orientation. [Hint: Exercise 5.4.6]
2. Suppose $A, B$ are hyperbolic and have the same orientation.
   (a) Show that $\mathbb{R} \setminus \{0\}$ is invariant for both A and B.
   (b) Use Exercises 5.4.2 and 5.4.4 to construct a conjugacy $h$ between $A|_{\mathbb{R}\setminus\{0\}}$ and $B|_{\mathbb{R}\setminus\{0\}}$.

(c) Extend $h$ to $\mathbb{R}$ by letting $h(0) = 0$ and show that $h$ is a conjugacy between A and B.

**Exercise 5.4.6.** Show that conjugacy preserves invariant sets, i.e. if $f : X \to X, g : Y \to Y$ are invertible maps and $h : X \to Y$ is a conjugacy between $f$ and $g$, then $X' \subseteq X$ is invariant for $f$ if and only if $Y' := h(X')$ is invariant for $g$.

**Exercise 5.4.7.** Consider the setting of Exercise 5.4.4 in the special case where $X = Y = \mathbb{R}$, $f, g$ are linear maps $A(x) = ax, B(x) = bx$ with $a, b \notin \{0, \pm 1\}$ and with the same orientation, $X' = Y' = \mathbb{R} \setminus \{0\}$, and $\mathcal{U}, \mathcal{V}$ are the fundamental domains given by Exercises 5.4.2 and 5.4.3.

1. Show that the there exists a homeomorphism $\tilde{h} : \mathcal{U} \to \mathcal{V}$;
2. Show that $h : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$, defined as in (5.1), is a homeomorphism. [*Hint:* consider first the order-preserving case for simplicity and show that $h$ is continuous, the continuity of $h^{-1}$ is proved in the same way. A function of one variable is continuous if it is continuous at every point $x$ and it is continuous at $x$ if it is continuous from the left and from the right. Now distinguish two cases: *i)* if $x \in \mathbb{R} \setminus \{0\}$ is such that $A^{\tau(x)}(x)$ lies in the interior of $\mathcal{U}$ then both are easy; *ii)* if $x$ is such that $A^{\tau(x)}(x)$ lies on the boundary of $\mathcal{U}$ then continuity either from the left or from the right is easy, for the not easy case, you will need to use the $\epsilon, \delta$ definition of continuity
3. Extend $h$ to all of $\mathbb{R}$ by letting $h(0) = 0$. Show that this defines a global homeomorphism $h : \mathbb{R} \to \mathbb{R}$ if and only if A, B are either both contracting or both expanding. [*Hint:* show that $h$ is continuous at 0]

**Exercise 5.4.8.** Prove Theorem 5.8 assuming Theorem 5.2.

# Chapter 6

# Interval Diffeomorphisms

In this chapter we begin the study of nonlinear maps. While linear maps have the characteristic that they essentially look the same at every scale, the *global* dynamics of non linear maps can be very complicated, partly because nonlinear maps can have many fixed and periodic points which interact in subtle ways. We will therefore begin our study of non-linear maps with some simple situations, starting with diffeomorphisms of an interval.

Throughout this chapter we suppose that $I = [a, b] \subset \mathbb{R}$ is a compact interval and $f : I \to I$ is a $C^1$ diffeomorphism of $I$. In particular this means that $f'(x) \neq 0$ for all $x \in I$. Thus either $f'(x) > 0$ for all $x \in I$ or $f'(x) < 0$ for all $x \in I$.

**Definition 6.1.** Let $f : I \to I$ is a $C^1$ diffeomorphism. If $f'(x) > 0$ for all $x \in I$ we say that $f$ is *orientation preserving* and otherwise that it is *orientation reversing.*

*Remark* 6.2. If $f$ is orientation reversing, then $f^2$ is orientation preserving, and so are all forward and backward iterates of $f^2$. Thus we can always (almost) reduce the situation to the orientation preserving case.

## 6.1 Fixed points and limit sets

**Lemma 6.3.** *Let $f : I \to I$ be a $C^1$ diffeomorphism of a compact interval. Then $f$ maps endpoints to endpoints. In particular, $f$ has at least one fixed point and if $f$ is orientation preserving then it has at least two fixed points and these are the endpoints of the interval.*

*Proof.* Exercise 6.5.1. □

**Lemma 6.4.** *Let $f : I \to I$ be a $C^1$ orientation preserving diffeomorphism of a compact interval. Then for any $x_0 \in I$ the limit sets $\alpha(x_0), \omega(x_0)$ are fixed points.*

*Proof.* Exercise 6.5.2 □

**Definition 6.5.** A fixed point $p \in I$ is *attracting* if there exists a neighbourhood $\mathcal{U}$ of $p$ such that $\omega(x_0) = \{p\}$ for all $x_0 \in \mathcal{U}$; the neighbourhood $\mathcal{U}$ is called the *local basin of attraction* of $p$. A fixed point $p \in I$ is *repelling* if there exists a neighbourhood $\mathcal{U}$ of $p$ such that $\alpha(x_0) = \{p\}$ for all $x_0 \in \mathcal{U}$, i.e. it is attracting for $f^{-1}$ with local basin of attraction $\mathcal{U}$.

*Remark* 6.6. Notice that points may be neither attracting nor repelling, e.g. the identity map.

## 6.2 Hyperbolic fixed points

Let $f : I \to I$ be a $C^1$ interval diffeomorphism.

**Definition 6.7.** A fixed point $p \in I$ is *hyperbolic* if $|f'(p)| \neq 1$.

**Lemma 6.8.** *Let $p$ be a hyperbolic fixed point. Show that if $|f'(p)| < 1$ then $p$ is attracting and if $|f'(p)| > 1$ then $p$ is repelling.*

*Proof.* Exercise 6.5.6. □

Non-hyperbolic fixed points can have a variety of behaviours as can be easily seen by sketching a few graphs. Indeed the attracting or repelling nature of a fixed points can easily be interpreted geometrically by the shape of the graph of $f$ in a neighbourhood of the fixed point, see Exercise 6.5.8, and this in turn can be expressed in terms of higher order derivatives of $f$ at $p$, see Exercise 6.5.9.

**Definition 6.9.** $f$ is *hyperbolic* if every fixed point of $f$ is hyperbolic.

**Lemma 6.10.** *If $f$ is hyperbolic then it has a finite number of fixed points*

*Proof.* Exercise 6.5.10. □

*Remark* 6.11. Although we will not define this notion precisely here, hyperbolic fixed points are "generic" in the sense that they remain hyperbolic under small perturbations and non-hyperbolic fixed points can be made hyperbolic by arbitrarily small perturbations.

## 6.3 Conjugacy classes

We start by considering the case in which $f$ has only two fixed points.

**Proposition 6.12.** *Let $f, g$ be $C^1$ orientation preserving interval diffeomorphisms, each with exactly two fixed points . Then $f, g$ are topologically conjugate.*

*Proof.* Exercise 6.5.11 □

This shows that all $C^1$ diffeomorphisms with two fixed points belong to the same topological conjugacy class. They cannot be $C^1$ conjugate unless the derivatives at the fixed points are the same. We can now state perhaps the most important result of this section.

**Theorem 6.13.** *Two orientation-preserving hyperbolic interval diffeomorphisms $f, g$ are topologically conjugate if and only if they have the same number of attracting and the same number of repelling fixed points.*

*Proof.* One direction is clear. Since topological conjugacies preserve fixed points and limit sets, if two interval diffeomorphisms have different numbers of attracting and/or repelling fixed points they cannot be topologically conjugate. For the other direction, notice first that attracting and repelling fixed points must alternate on the interval. Each closed subinterval between one attracting and one repelling fixed point is invariant and can itself be considered an interval diffeomorphism with exactly two fixed points, and therefore we can apply Lemma 6.12 to get a conjugacy restricted to the corresponding intervals. Gluing together these conjugacies we get a global conjugacy. □

It is easy to see that the result fails without the hyperbolicity assumption, the number of fixed points does not in general characterise completely the topological conjugacy class, see Exercise 6.5.12. Clearly it continues to hold that $f, g$ cannot be $C^1$ conjugate if they do not have the same derivatives at corresponding fixed points.

## 6.4 Structural stability

To study the bifurcations and structural stability of interval diffeomorphisms we need to introduce a topology on the space of all $C^1$ diffeomorphisms of the interval $I$. We shall in fact introduce two natural metrics on the spaces $C^0(I)$ of all continuous maps $f : I \to I$ and on the space $C^1(I)$ of all $C^1$ maps $f : I \to I$. For $f, g \in C^0(I)$ we let

$$d_0(f, g) = \sup_{x \in I}\{|f(x) - g(x)|\}.$$

For $f, g \in C^1(I)$ we let

$$d_1(f, g) := \sup_{x \in I}\{|f(x) - g(x)| + |f'(x) - g'(x)|\}.$$

*Remark* 6.14. Notice that the definitions do not require $f, g$ to be invertible.

Notice that $d_1(f,g) \geq d_0(f,g)$. Indeed we can have two maps which are arbitrarily close in the $C^0$ metric and far in the $C^1$ metric.

*Example* 20. Suppose $I = [0,1]$ and let $f : [0,1] \to [0,1]$ and $g : [0,1] \to [0,1]$ be

$$f(x) \equiv \frac{1}{2} \quad \text{and} \quad g(x) = \frac{1}{2} + \epsilon \sin \frac{x}{\epsilon}$$

Notice that the graph of $g$ is contained in an $\epsilon$-neighbourhood of the graph of $f$ and therefore $d_0(f,g) = \epsilon$ However we have $Df(x) \equiv 0$ and $Dg(x) = \cos x$ and therefore $d_1(f,g) = 1$. Thus the maps $f$ and $g$ are very close in the $C^0$ metric for $\epsilon$ small, but always far in the $C^1$ metric.

This means that the two metrics induce different topologies: *you can have a sequence of functions $f_n$ converging to $f$ in the $C^0$ metric but not in the $C^1$ metric* and therefore a "small" perturbation in the $C^0$ metric may be "large" in the $C^1$ metric. We thus have two different equivalence relations and two different topologies and we want to study the problem of structural stability with respect to the different combinations of topologies and equivalence classes. We fix each of the topologies in turn and consider the two possible equivalence classes.

**Proposition 6.15.** *Let $f$ be a hyperbolic $C^1$ interval diffeomorphism. Then*
*1) $f$ is $C^1$ structurally stable with respect to topological conjugacy;*
*2) $f$ is $C^0$ structurally unstable with respect topological conjugacy;*
*3) $f$ is $C^1$ structurally unstable with respect to $C^1$ conjugacy;*
*4) $f$ is $C^0$ structurally unstable with respect to $C^1$ conjugacy.*

*Proof.* 1) It is sufficient to show that for some sufficiently small $\epsilon > 0$, all $g$ with $d_1(f,g) < \epsilon$ are hyperbolic and have the same number of fixed points as $f$, since this implies that $f, g$ are topologically conjugate. We argue as follows.

Since $f$ is hyperbolic all its fixed points are isolated and there exist constants $\delta, \delta' > 0$ such that, letting $\mathcal{U}_p = [p-\delta, p+\delta]$ denote a neighbourhood of each fixed point $p$, we have:

(i) $\mathcal{U}_p \cap \mathcal{U}_q = \emptyset$ if $p \neq q$ ;
(ii) $|f(p) - f(p \pm \delta)| \geq \delta'$ for every fixed point $p$;
(iii) $|f'(x)| \neq 1$ for all $x \in \mathcal{U}_p$ for every fixed point $p$.

Condition (2) implies that as long as $g$ is sufficiently $C^0$ close to $f$, it has no fixed point outside the union of the neighbourhoods $\mathcal{U}_p$. Thus we just need to show that as long as $g$ is sufficiently $C^1$ close to $f$ then there exists a unique hyperbolic fixed point inside each neighbourhood $\mathcal{U}_p$.

It is clear that there must be at least one fixed point since the images of $f(p \pm \delta)$ lie on opposite sides of the diagonal and so the same must be true of $g$. To see that it is unique, suppose by contradiction that there exist two fixed points $q, q' \in \mathcal{U}_p$. Then we would have $|f(q) - f(q')| = |q - q'|$ and thus by the mean value theorem there would exist some point $x \in [q, q'] \subseteq \mathcal{U}_p$ such that $f'(x) = 1$, contradicting (iii).

2) The second part of the argument above does not hold if the perturbation is small only in the $C^0$ metric, Clearly we can perturb $f$ inside the neighbourhood $U_p$ to obtain $g$ with $g(\mathcal{U}_p) \subseteq (p - \epsilon, p + \epsilon)$ but so that $g$ has two fixed points in $\mathcal{U}_p$. Thus $g$ cannot be topologically conjugate to $f$.

3) It is clear that it is always possible to find another diffeomorphism $g$ such that the derivatives at the fixed point of $g$ do not coincide with the derivatives at the corresponding fixed points of $f$. Thus, even if $f, g$ have the same number of fixed points, they are not $C^1$ conjugate.

4) Is a a trivial consequence of 2). □

## 6.5 Exercises

**Exercise 6.5.1.** Prove that a $C^1$ interval diffeomorphism must send endpoints to endpoints. Conclude that every interval diffeomorphism has at least one fixed point and if $f$ is orientation preserving it has at least two fixed points and these are the endpoints of the interval.

**Exercise 6.5.2.** Let $f : I \to I$ be a $C^1$ orientation preserving diffeomorphism of a compact interval. Show that any orbit $\mathcal{O}^+(x_0)$ is a monotone sequence in $I$. Conclude that $\omega(x_0)$ is a fixed point. Deduce also that $\alpha(x_0)$ is a fixed point. [*Hint: use the Mean Value Theorem*]

**Exercise 6.5.3.** Let $f : I \to I$ be a $\mathcal{C}^1$ orientation-reversing diffeomorphism of a compact interval. Show that, for any $x_0 \in I$, the sets $\alpha(x_0)$ and $\omega(x_0)$ are fixed points or points of period two. *Hint: Consider $f^2$.*

**Exercise 6.5.4.** (1) Find the fixed points of the interval map

$$f : [0, 1] \to [0, 1]; \quad x \mapsto \frac{x + x^2}{2}.$$

(2) Are these fixed points repelling or attracting? (3) What can you say about $\omega(x_0)$ and $\alpha(x_0)$ for any $x_0 \in [0, 1]$?

**Exercise 6.5.5.** Find an explicit formula for a $\mathcal{C}^1$ interval diffeomorphism $f : I \to I$ which has exactly three fixed points.

**Exercise 6.5.6.** Let $p$ be a hyperbolic fixed point. Show that *i)* if $|f'(p)| < 1$ then $p$ is attracting; *ii)* if $|f'(p)| > 1$ then $p$ is repelling.

**Exercise 6.5.7.** Sketch the graph of an interval diffeomorphism with an attracting fixed point $p$ with $f'(p) \in (-1, 0)$. Sketch the first few iterates of an initial condition $x_0$ in its immediate basin of attraction.

**Exercise 6.5.8.** Sketch the graph of interval diffeomorphisms with a fixed point $p$ with $f'(p) = 1$ and such that $p$ is: *i)* attracting; *ii)* repelling; *iii)* attracting on one side and repelling on the other.

**Exercise 6.5.9.** Let $f : I \to I$ be a $C^3$ orientation preserving diffeomorphism of a compact interval and $p \in (0, 1)$ a fixed point with $f'(p) = 1$. Give conditions on the higher order derivatives of $f$ at $p$ to guarantee that $p$ is *i)* attracting; *ii)* repelling.

**Exercise 6.5.10.** Show that every hyperbolic interval diffeomorphism has a finite number of fixed points. [*Hint: show that any accumulation point of the set of fixed points is a fixed point. Then argue by contradiction.* ]

**Exercise 6.5.11.** Let $f, g$ be $C^1$ orientation preserving interval diffeomorphisms, each with exactly two fixed points.

1. Show that both maps have one attracting and one repelling fixed point;
2. find fundamental domains for the interior of the intervals;
3. use these to construct conjugacies between $f$ and $g$.

**Exercise 6.5.12.** Sketch the graphs of two orientation-preserving $\mathcal{C}^1$ interval diffeomorphisms with the same number of fixed points, but which are not topologically conjugate, and explain why they are not topologically conjugate.

**Exercise 6.5.13.** For $f, g \in \mathcal{C}^0(I, I)$ let $d_0(f, g) = \sup_{x \in I}\{|f(x) - g(x)|\}$. For $f, g \in \mathcal{C}^1(I, I)$ let $d_1(f, g) = \sup_{x \in I}\{|f(x) - g(x)| + |f'(x) - g'(x)|\}$. Show that $d_0$ $d_1$ are metrics on $\mathcal{C}^0(I, I)$ and $\mathcal{C}^1(I, I)$ respectively.

# Part III

# Local Dynamics

# Chapter 7

# Two-Dimensional Linear Maps: Real Eigenvalues

We recall that a map $A : \mathbb{R}^n \to \mathbb{R}^n$ is *linear* if

$$A(\alpha v + \beta w) = \alpha A(v) + \beta A(w)$$

for any vectors $v, w \in \mathbb{R}^n$ and any scalars $\alpha, \beta \in \mathbb{R}$. Any such map can be represented by an $n \times n$ matrix which for convenience we identify with A. We say that $\lambda$ is an *eigenvalue* of $A$ if it is a solution to the equation $det(A - \lambda I) = 0$. Eigenvalues can be real or complex. The set of eigenvalues of $A$ is called the *spectrum* of $A$. We introduce the following terminology.

**Definition 7.1.** (1) A is *invertible* if $\det A \neq 0$;
(2) A is *orientation-peserving* if $\det A > 0$ and *orientation-reversing* if $\det A < 0$.
(3) $A$ is *hyperbolic* if $|\lambda| \neq 1$ for all eigenvalues $\lambda$ of A.

For simplicity, in these notes we will restrict our attention to the case of two-dimensional linear maps, since such class of maps is already sufficiently rich to introduce many new ideas, and the higher dimensional results can often be obtained by technical but rather straightforward generalizations of the two-dimensional case. A two-dimensional linear map can be represented by a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$A$ has two eigenvalues (counted with multiplicity) given by solutions of the characteristic equation which in this case gives $\lambda^2 - (a + d)\lambda + (ad - bc) = 0$ and therefore

$$\lambda_{1,2} = \frac{a + d \pm \sqrt{(a + d)^2 - 4(ad - bc)}}{2}$$

It follows immediately that the eigenvalues are either both real or a pair of complex conjugate eigenvalues. In this section we discuss the case in which the eigenvalues are real, in the next we will consider the case of complex eigenvalues.

## 7.1 Real eigenvalues

In this chapter we focus on the case in which the eigenvalues of $A$ are real. We start by studying the special case in which the matrix representing $A$ is diagonal, and then consider the general case.

### 7.1.1 Diagonal matrices

We say that the matrix $A$ is *diagonal* if it has the form

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Then, for every $n \in \mathbb{Z}$ we have

$$A^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}$$

Therefore, for any vector $v_0 = (v_0^{(1)}, v_0^{(2)}) \in \mathbb{R}^2$ which we think of as an initial condition we have

$$\begin{pmatrix} v_n^{(1)} \\ v_n^{(2)} \end{pmatrix} = A^n \begin{pmatrix} v_0^{(1)} \\ v_0^{(2)} \end{pmatrix} = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \begin{pmatrix} v_0^{(1)} \\ v_0^{(2)} \end{pmatrix} = \begin{pmatrix} \lambda_1^n v_0^{(1)} \\ \lambda_2^n v_0^{(2)} \end{pmatrix}$$

From this, it is very straightforward to analyse the possible cases. We leave this is an exercise.

**Definition 7.2.** A linear map with real distinct eigenvalues $\lambda_{1,2}$ is *hyperbolic* if $|\lambda_{1,2}| \neq 1$. A hyperbolic linear map $A$ has a unique fixed point at the origin. This fixed point is called

1. a *sink* if $|\lambda_1| < 1$ and $|\lambda_2| < 1$
2. a *source* if $|\lambda_1| > 1$ and $|\lambda_2| > 1$
3. a *saddle* if $|\lambda_1| < 1 < |\lambda_2|$ or $|\lambda_2| < 1 < |\lambda_1|$.

### 7.1.2 General matrices

For a general matrix A (not necessarily in diagonal form) with real eigenvalues $\lambda_1, \lambda_2$ we can define the corresponding eigenspaces $E^{(1)}, E^{(2)}$ which are characterised by the property that

$$Av^{(1)} = \lambda_1 v^{(1)} \quad \text{and} \quad Av^{(2)} = \lambda_2 v^{(2)}$$

for any vectors $v^{(1)} \in E^{(1)}, v^{(2)} \in E^{(2)}$. In particular, this means that the eigenspaces are *invariant* under the action of $A$ and of $A^{-1}$:

$$A(E^{(1)}) = A^{-1}(E^{(1)}) = (E^{(1)}) \quad \text{and} \quad A(E^{(2)}) = A^{-1}(E^{(2)}) = (E^{(2)})$$

and that the action of A on these eigenspaces is simply that of the one-dimensional linear maps $v^{(1)} \mapsto \lambda_1 v^{(1)}$ and $v^{(2)} \mapsto \lambda_2 v^{(2)}$ respectively. The dynamics of $A$ restricted to each of these eigenspaces can therefore be classified according to the values of $\lambda_1, \lambda_2$ exactly as in the one-dimensional case. Notice moreover, that this is also the same as in the diagonal case which is just a special case in which the eigenspaces correspond to the horizontal and vertical axes.

Notice that this implies in particular that if $\lambda_1 \neq \lambda_2$ then also $E^{(1)} \neq E^{(2)}$. If $\lambda_1 = \lambda_2$ then it may happen that $E^{(1)} \neq E^{(2)}$ or it may happen that $E^{(1)} = E^{(2)}$. The case in which $E^{(1)} = E^{(2)}$ is more complicated to analyze and therefore for simplicity we shall often assume that A has two distinct eigenspaces, or the stronger condition that it has distinct eigenvalues. In this case, since $E^{(1)}, E^{(2)}$ are linearly independent and span $\mathbb{R}^2$, any vector $v_0$ can be written in a unique way as

$$v_0 = v_0^{(1)} + v_0^{(2)}$$

for some vectors $v_0^{(1)} \in E^{(1)}$ and $v_0^2 \in E^{(2)}$. By linearity we then have

$$v_n = A^n v_0 = A(v_0^{(1)} + v_0^{(2)}) = Av_0^{(1)} + Av_0^{(2)} = \lambda_1^n v_0^{(1)} + \lambda_2^n v_0^{(2)}$$

Once again, therefore, it is easy to analyse the possible dynamical configurations depending on the values of $\lambda_1$ and $\lambda_2$.

## 7.2 Linear conjugacy

We can formalise these observations by saying that the matrix A is equivalent to a matrix B in diagonal form through a *linear change of coordinates*.

**Definition 7.3.** Two invertible matrices A, B are *linearly conjugate* if there exists an invertible matrix P such that

$$AP = PB.$$

Notice that if $A, B$ are linearly conjugate, then for any $n \in \mathbb{Z}$ we have

$$A^n = (PBP^{-1})^n = PBP^{-1}PBP^{-1} \cdots PBP^{-1}PBP^{-1} = PB^nP^{-1}.$$

Therefore, the conjugacy is actually a conjugacy of the *dynamical systems* given by the iterates of the matrices $A, B$.

**Proposition 7.4.** *Suppose $A$, $B$ are $2 \times 2$ matrices, both having the same pair of distinct real eigenvalues $\lambda_1, \lambda_2$. Then $A$, $B$ are linearly conjugate. Conversely, suppose $A, B$ are linearly conjugate, then they have the same eigenvalues.*

*Proof.* Suppose first that $A, B$ have the same eigenvalues. It is sufficient to show that they are both conjugate to the same matrix in diagonal form. We show it for A, obviously the same argument applies to $B$. Let $e^{(1)} = (v^{(1)}, v^{(2)}), e^{(2)} = (w^{(1)}, w^{(2)})$ be eigenvectors of $A$ corresponding to $\lambda_1, \lambda_2$ respectively. Let

$$P = \begin{pmatrix} v^{(1)} & w^{(1)} \\ v^{(2)} & w^{(2)} \end{pmatrix}$$

We leave it as an exercise to show that

$$Az = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1} z$$

proving that $P$ is a linear conjugacy (Exercise 7.4.1). □

# 7.3 Topological Conjugacy

We are now ready to classify all two dimensional invertible and hyperbolic linear maps in two dimensions with real distinct eigenvalues. In this case there are three possibilities for the fixed point: it can be *attracting*, *repelling*, or it can be a *saddle point*, corresponding to the various possibilities depending on whether the eigenvalues have modulus less than 1 or greater than 1. Moreover, the linear map can be *orientation preserving* if $\det A > 0$ or *orientation reversing* if $\det A < 0$.

**Theorem 7.5.** *Let $A, B$ be two hyperbolic invertible two-dimensional linear maps with real distinct eigenvalues $\lambda_1^A < \lambda_2^A$ and $\lambda_1^B < \lambda_2^B$ respectively. Suppose that*
*(1) A and B have the same kind of fixed point (attracting, repelling, saddle-type);*
*(2) A and B are either both orientation preserving or orientation reversing;*
*(3) if the fixed points are of saddle type, suppose that the sign of the corrresponding (contracting/repelling) eigenvalues of A and B are the same.*
*Then A and B are topologically conjugate.*

*Proof.* It is sufficient to consider the case in which both $A$ and $B$ are in diagonal form. Indeed by Lemma 7.4, A and B are both linearly (and thus in particular, topologically) conjugate to linear maps $\tilde{A}, \tilde{B}$ in diagonal form with the same eigenvalues as $A$ and $B$ respectively. Thus, A, B are toplogically conjugate if and only if $\tilde{A}, \tilde{B}$ are topologically conjugate. We assume therefore that both $A$ and $B$ are in diagonal form

$$A = \begin{pmatrix} \lambda_1^A & 0 \\ 0 & \lambda_2^A \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \lambda_1^B & 0 \\ 0 & \lambda_2^B. \end{pmatrix}$$

In particular in both cases the coordinate axes are invariant and the dynamics on the axes are just given by the one-dimensional linear maps $x_1 \mapsto \lambda_1^A x_1$, $x_2 \mapsto \lambda_2^A x_2$, $y_1 \mapsto \lambda_1^B y_1$ and $y_2 \mapsto \lambda_2^B y_2$. Then we can apply the results for the conjugacy of one-dimensional maps to show that the corresponding axes are topologically conjugate, i.e. there exist homeomorphisms $h_1 : \mathbb{R} \to \mathbb{R}$ and $h_2 : \mathbb{R} \to \mathbb{R}$ such that

$$h_1(\lambda_1^A(x_1)) = \lambda_1^B(h_1(x_1)) \quad \text{and} \quad h_2(\lambda_1^B(x_2)) = \lambda_2^B(h_2(x_2)) \tag{7.1}$$

for every $(x_1, x_2) \in \mathbb{R}^2$. We then define $h : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$h(x_1, x_2) = (h_1(x_1), h_2(x_2)) \tag{7.2}$$

for any $x = (x_1, x_2) \in \mathbb{R}^2$. Then clearly $h$ is a homeomorphism. We leave it as Exercise 7.4.2 to show that this is a conjugacy between the linear maps $A, B$. □

Finally we show that linear maps are not generally $C^1$ conjugate unless they are linearly conjugate.

**Proposition 7.6.** *Suppose that $f, g : \mathbb{R}^n \to \mathbb{R}^n$ $C^1$ maps with fixed point $p, q$ respectively. Suppose $f, g$ are conjugate by a $C^1$ diffeomorphism $h$ and $h(p) = q$. Then $Df_p$ and $Dg_q$ have the same eigenvalues.*

*Proof.* By the definition of conjugacy we have $f = h^{-1} \circ g \circ h$. Differentiating both sides, by the chain rule, for any $x$ we have $Df_x = Dh^{-1}_{g(h(x))} \circ Dg_{h(x)} \circ Dh_x$. Letting $x = p$ and using the fact that $h(x) = q$ is a fixed point for $g$ this gives

$$Df_p = Dh^{-1}_{g(h(p))} \circ Dg_{h(p)} \circ Dh_p = Dh_q^{-1} \circ Dg_q \circ Dh_p$$

Since $q = h(p)$, we have $Dh_q^{-1} = (Dh_p)^{-1}$ and therefore $Df_p = Dh_p^{-1} \circ Dg_q \circ Dh_p$. Thus $Df_p$ and $Dg_q$ are linearly conjugate by the linear map $Dh_p$ and so in particular have the same eigenvalues. □

## 7.4 Exercises

**Exercise 7.4.1.** $A, B$ have the same eigenvalues real distinct eigenvalues $\lambda_1, \lambda_2$ with eigenvactors $e^{(1)} = (v^{(1)}, v^{(2)}), e^{(2)} = (w^{(1)}, w^{(2)})$. Let

$$P = \begin{pmatrix} v^{(1)} & w^{(1)} \\ v^{(2)} & w^{(2)} \end{pmatrix}$$

Show that $P$ maps the eigenvectors $e^{(1)}, e^{(2)}$ to unit horizontal and vertical vectors. Now let $z = z^{(1)}e^{(1)} + z^{(2)}e^{(2)} \in \mathbb{R}^2$ be an arbitrary vector. Show that $Az = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1} z$.

**Exercise 7.4.2.** Show that $h$, as defined in (7.2) is a conjugacy between $A, B$.

# Chapter 8

# Two-Dimensional Linear Maps: Complex Eigenvalues

We now turn to the study of linear systems with complex eigenvalues. Once again we start with a matrix in a "normal form" which corresponds to the diagonal matrices for the case of real eigenvalues.

## 8.1 Complex Eigenvalues

### 8.1.1 Normal form for complex eigenvalues

We start with the special case of maps of the form

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

Identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ by the identification $(x, y) \leftrightarrow x + iy$. Then the action of $A$ corresponds exactly to multiplication by the complex number $\alpha - i\beta$. Indeed:

$$A(x, y) = (\alpha x+\beta y, -\beta x+\alpha y) \quad \text{and} \quad (x+iy)(\alpha-i\beta) = (\alpha x+\beta y)+i(-\beta x+\alpha y).$$

Writing the complex number $\alpha + i\beta$ in polar coordinates we get

$$\alpha - i\beta = re^{i\theta} \quad \text{where} \quad r = |\alpha - i\beta| \text{ and } \theta = \cos^{-1}(\alpha/r).$$

Then, writing an initial condition in polar form as $r_0e^{i\theta_0}$ we then get the iterate

$$A(r_0e^{i\theta_0}) = re^{i\theta}r_0e^{i\theta_0} = rr_0e^{i(\theta_0+\theta)}$$

and more generally

$$A^n(r_0e^{i\theta_0}) = r_0r^ne^{i(\theta_0+n\theta)}.$$

From this we can easily classify all the possibilities in the hyperbolic case: if $r < 1$ then all trajectories spiral in towards the origin, if $r > 1$ then they spiral out towards infinity. Notice moreover that the direction of the spiral depends on the sign of $\theta$ which in turn depends on the sign of $\alpha$, i.e. whether the eigenvalues have positive or negative real part.

### 8.1.2 Linear conjugacy

The matrix we studied above was of a very special form but it is representative of all matrices with complex eigenvalues in a similar way to the way diagonal matrices are representative of all matrices with real aigenvalues.

**Proposition 8.1.** *Any matrix $B$ with complex conjugate eigenvalues $\alpha \pm i\beta, \beta \neq 0$ is linearly conjugate to the matrix*

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

*Proof.* Even though the entries of $B$ are real, we can think of it as a complex matrix, defining a map $B : \mathbb{C}^2 \to \mathbb{C}^2$ and, exactly as in the real case, the eigenvalues $\alpha \pm i\beta$ define eigenvectors $e^{(1)} \pm ie^{(2)}$ where $e^{(1)}, e^{(2)} \in \mathbb{R}^2$. Let us write the vectors $e^{(1)} = (v^{(1)}, v^{(2)}), e^{(2)} = (w^{(1)}, w^{(2)})$ in standard Euclidean coordinates. Then just as we did before, we can define the matrix

$$P = \begin{pmatrix} v^{(1)} & w^{(1)} \\ v^{(2)} & w^{(2)} \end{pmatrix}$$

We claim that $B = PAP^{-1}$. Indeed, as above we have that

$$P \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix}; \quad P^{-1} \begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad P \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} w^{(1)} \\ w^{(2)} \end{pmatrix}; \quad P^{-1} \begin{pmatrix} w^{(1)} \\ w^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Now let $z = z_1 e^{(1)} + z_2 e^{(2)}$ be an arbitrary vector which we choose, for convenience, to write as a linear combination of eigenvectors. Then

$$P^{-1}(z) = P^{-1}(z_1 e^{(1)} + z_2 e^{(2)}) = P^{-1}(z_1 e^{(1)}) + P^{-1}(z_2 e^{(2)}) = \begin{pmatrix} z_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

Therefore

$$AP^{-1}(z) = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \alpha z_1 + \beta z_2 \\ -\beta z_1 + \alpha z_2 \end{pmatrix}$$

and thus

$$PAP^{-1}(z) = \begin{pmatrix} v^{(1)} & w^{(1)} \\ v^{(2)} & w^{(2)} \end{pmatrix} \begin{pmatrix} \alpha z_1 + \beta z_2 \\ -\beta z_1 + \alpha z_2 \end{pmatrix} = \begin{pmatrix} v^{(1)}\alpha z_1 + v^{(1)}\beta z_2 - w^{(1)}\beta z_1 + w^{(1)}\alpha z_2 \\ v^{(2)}\alpha z_1 + v^{(2)}\beta z_2 - w^{(2)}\beta z_1 + w^{(2)}\alpha z_2 \end{pmatrix}$$

On the other hand, by the definition of eigenvector, we have

$$B(e^{(1)} + ie^{(2)}) = (\alpha + i\beta)(e^{(1)} + ie^{(2)}) = \alpha e^{(1)} - \beta e^{(2)} + i(\beta e^{(1)} + \alpha e^{(2)})$$

and therefore, equating real and immaginary parts, we get

$$Be^{(1)} = \alpha e^{(1)} - \beta e^{(2)} \quad \text{and} \quad Be^{(2)} = \beta e^{(1)} + \alpha e^{(2)}.$$

From this we have

$$Bz = B(z_1 e^{(1)} + z_2 e^{(2)}) = z_1 \alpha e^{(1)} - z_1 \beta e^{(2)} + z_2 \beta e^{(1)} + z_2 \alpha e^{(2)}.$$

Since $e^{(1)} = (v^{(1)}, v^{(2)})$ and $e^{(2)} = (w^{(1)}, w^{(2)})$ we get $Bz = PAP^{-1}z$ and thus completes the proof. □

## 8.2 Topological conjugacy II

In Section 7.3 we studied the equivalence relation of topological conjugacy amongst hyperbolic linear maps with real distinct eigenvalues. Here we carry out a similar study for maps with complex conjugate eigenvalues. We show that not only are many of such maps topologically conjugate to each other, but that many maps with complex conjugate eigenvalues are also conjugate to corresponding maps with real distinct eigenvalues.

**Proposition 8.2.**

More specifically we recall that all hyperbolic linear maps with distinct eigenvalues fall into one of three categories depending on the nature of the fixed point at the origin. The fixed point is *attracting* if both eigenvalues have real part with modulus $< 1$, *repelling* if both eigenvalues have real part with modulus $> 1$ and a *saddle point* if one eigenvalue has real part with modulus $< 1$ and the other eigenvalue has real part with modulus $> 1$ (in which case of course both eigenvalues have to be real).

**Theorem 8.3.** *Two hyperbolic invertible linear maps with distinct (real or complex) eigenvalues are topologically conjugate if and only their fixed points are both attracting, or both repelling or both saddle points,* ***and*** *if they have the same number of eigenvalues with positive real part.*

*Proof.* Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

be two invertible hyperbolic matrices with distinct eigenvalues $\lambda_1, \lambda_2$ and $\lambda_1', \lambda_2'$ respectively. If they are both saddle points then all eigenvalues must be real and the result has already been proved above. Thus we can assume the fixed points are either both attracting, corresponding to the case $\max\{|\lambda_1|, |\lambda_2|, |\lambda_1'|, |\lambda_2'|\} < 1$ or repelling, corresponding to the case $\min\{|\lambda_1|, |\lambda_2|, |\lambda_1'|, |\lambda_2'|\} > 1$. For definiteness. let us suppose that both fixed points are attracting, the repelling case follows by the same arguments. Let us suppose moreover that the eigenvalues of $A$ are real and the eigenvalues of $A'$ are complex. The other cases follow by essentially identical arguments. We remark first of all that, as we have seen, $A, A'$ are both linearly conjugate to corresponding matrices in "normal form", so we can assume without loss of generality that

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

with $0 < \lambda_1 < \lambda_2 < 1$ and $|\lambda_{1,2}'| = |\alpha \pm i\beta| < 1$. We will construct fundamental domains for the maps $A, B$. Let $\mathcal{S}$ denote the unit circle, then its image $A(\mathcal{S})$ is a smooth curve strictly contained inside the unit circle. Similarly $B(\mathcal{S})$ is a smooth curve strictly contained inside $\mathcal{S}$. Let $\mathcal{D}$ denote the annular region bounded by $\mathcal{S}$ and $A(\mathcal{S})$ and $D'$ denote the annular region bounded by $\mathcal{S}$ and $B(\mathcal{S})$ (in each case we include $\mathcal{S}$ in $\mathcal{D}, \mathcal{D}'$ but not its images $A(\mathcal{S})$ and $B(\mathcal{S})$. We start by showing that $\mathcal{D}$ and $\mathcal{D}'$ are fundamental domains for the corresponding maps.

**Lemma 8.4.** $\forall\, x \in \mathbb{R}^2 \setminus \{0\}$ *there exists a unique* $\tau(x) \in \mathbb{Z}$ *such that* $A^{\tau(x)}(x) \in \mathcal{D}$. $\forall\, y \in \mathbb{R}^2 \setminus \{0\}$ *there exists a unique* $\tau'(y) \in \mathbb{Z}$ *such that* $B^{\tau'(y)}(y) \in \mathcal{D}'$.

*Proof.* The argument is essentially the same as that used above in the one dimensional case. The same argument works for $A$ and $B$ so for simplicity we just consider $A$. Each point $x \in \mathbb{R}^2$ lies on a smooth curve $\gamma(x)$ which is invariant under $A$ and which is monotone in the sense that for each constant $c > 0$ the intersection of $\gamma$ with the circle $\{|x| = c\}$ of radius $c$ centred at the origin is a unique point. Moreover, the orbit of every point $x$ is also monotone along $\gamma(x)$ in the sense that $|A^n(x)| \to 0$ monotonically as $n \to \infty$ and $|A^n(x)| \to \infty$ monotonically as $n \to -\infty$. It is therefore sufficient to show that for each $x \in \mathbb{R}^2 \setminus \{0\}$ there exists a unique $\tau(x)$ such that $A^{\tau(x)}(x) \in \gamma(x) \cap \mathcal{D}$. Suppose now for simplicity that $|x| > 1$, the case $|x| \leq 1$ is analogous. Let $\tau(x) \geq 1$ be the smallest positive integer such that $|A^{\tau(x)-1}(x)| > 1 \geq |A^{\tau(x)}(x)|$ Then, by the monotonicity of the orbits under $A$ we have $|A^{\tau(x)}(x)| > |A(1)| \geq |A^{\tau(x)+1}(x)|$ and all these points lie on the same curve $\gamma(x)$. This shows that $\tau(x)$ is the unique time in which the orbit of $x$ lands in $\mathcal{D}$. □

We now let $\overline{\mathcal{D}}, \overline{\mathcal{D}'}$ be the closures of $\mathcal{D}, \mathcal{D}'$.

**Lemma 8.5.** *There exists a homeomorphism* $\tilde{h} : \overline{\mathcal{D}} \to \overline{\mathcal{D}'}$ *such that for all* $x \in \mathcal{S}$

$$\tilde{h}(x) = B^{-1} \circ \tilde{h} \circ A(x)$$

*Proof.* The key observation here is to recall that all orbits of the linear maps $A, B$ lie on one of a family of invariant curves whose distance from the origin is monotonically decreasing. The intersection of these curves with the domains $\mathcal{D}, \mathcal{D}'$ gives a family of smooth curves with endpoints on the boundaries of $\mathcal{D}, \mathcal{D}'$, i.e. on $\mathcal{S}$ and $A(\mathcal{S}), B(\mathcal{S})$ respectively. To construct $\tilde{h}$ therefore we just need a homeomorphism which maps such a family of curves to each other. We start by defining $\tilde{h} : \mathcal{S} \to \mathcal{S}$ by an arbitrary homeomorphism and then map the corresponding curves to each other. This is clearly a bijection and a homeomoprhism since the curves are a continuous family, and gives the required conjugacy because $A(x)$ lies on the same curve as $x$ and therefore $\tilde{h}$ maps $\tilde{h}(x)$ to $B(\tilde{h}(x))$ by construction. □

We then define a map $h : \mathbb{R}^2 \to \mathbb{R}^2$ by letting $h(0) = 0$ and, for ever $x \in \mathbb{R}^2 \setminus \{0\}$ letting

$$h(x) := B^{-\tau(x)} \circ \tilde{h} \circ A^{\tau(x)}(x). \tag{8.1}$$

**Lemma 8.6.** *$h$ is a homeomorphism.*

*Proof.* The proof that $h$ is a bijection follows exactly as in the one-dimensional case and we omit the details. The proof that $h$ and $h^{-1}$ are continuous also follows by an almost identical argument as in the one-dimensional case. However we give some of the details in order also to clarify the role of Lemma 8.5 in the proof. We prove the continuity of $h$ as that of $h^{-1}$ follows by the same arguments.

Suppose that $x$ is such that $A^{\tau(x)}(x)$ lies in the interior of $\mathcal{D}$. Then, continuity at $x$ follows simply by the composition of locally continuous maps. The continuity at the origin also follows by exactly the same arguments as in the one-dimensional case. On the other hand, if $A^{\tau(x)}(x) \in \mathcal{S}$ then we need to take a little bit of care and to use the $\epsilon, \delta$ definition of continuity. We suppose $\epsilon > 0$ is fixed and seek $\delta > 0$ such that $|z - x| < \delta$ implies $|h(x) - h(z)| < \epsilon$. Notice that in this case the neighbourhood $|z - x| < \delta$ of $x$ is two-dimensional. For points within this neighbourhood we distinguish two possibilities: either $\tau(z) = \tau(x)$ or $\tau(z) > \tau(x)$. If $\tau(z) = \tau(x)$ then we can again use the continuity of $A, \tilde{h}, B$ to get the result. If $\tau(z) > \tau(x)$ then, supposing that $\delta > 0$ is sufficiently small, we have $\tau(z) = \tau(x) + 1$. Now consider the set

$$\mathcal{A} := \{z : |z - x| < \delta \text{ and } \tau(z) = \tau(x) + 1\}.$$

The set $\mathcal{A}$ is open, $A(\mathcal{A}) \subset \mathcal{D}$ and, for every $z \in \mathcal{A}$

$$h(z) = B^{-\tau(z)} \circ \tilde{h} \circ A^{\tau(z)}(z) = B^{-(\tau(x)+1)} \circ \tilde{h} \circ A^{\tau(x)+1}(z). \tag{8.2}$$

For any $\epsilon_0$ there exists $\delta > 0$ such that for all $z \in \mathcal{A}$ we have

$$z \in \mathcal{A} \quad \Longrightarrow \quad |A^{\tau(x)+1}(z) - A^{\tau(x)+1)}(x)| < \epsilon_0$$

Since $A(\mathcal{A}) \subset \mathcal{D}$ and $\tilde{h}$ is a homeomorphism, for any $\epsilon_1 > 0$ there exists $\epsilon_0 > 0$ such that

$$|A^{\tau(x)+1}(z) - A^{\tau(x)+1}(x)| < \epsilon_0 \quad \Longrightarrow \quad |\tilde{h} \circ A^{\tau(x)+1}(z) - \tilde{h} \circ A^{\tau(x)+1)}(x)| < \epsilon_1$$

Then, by the continuity of $B$, for any $\epsilon > 0$ there exists $\epsilon_1$ such that

$$|\tilde{h} \circ A^{\tau(x+1)}(z) - \tilde{h} \circ A^{\tau(x)+1)}(x)| < \epsilon_1$$

implies

$$|B^{-(\tau(x)+1)} \circ \tilde{h} \circ A^{\tau(x+1)}(z) - B^{-(\tau(x)+1)} \circ \tilde{h} \circ A^{\tau(x)+1)}(x)| < \epsilon \tag{8.3}$$

Now notice that

$$\begin{aligned} B^{-(\tau(x)+1)} \circ \tilde{h} \circ A^{\tau(x)+1)}(x) &= B^{-\tau(x)} \circ B^{-1} \circ \tilde{h} \circ A \circ A^{\tau(x)}(x) \\ &= B^{-\tau(x)} \circ \tilde{h} \circ A^{\tau(x)}(x) = h(x) \end{aligned}$$

by Lemma 8.5 and the fact that $A^{\tau(x)}(x) \in \mathcal{S}$. Thus, using this and (8.2) and substituting into (8.3) we get $|h(z) - h(x)| < \epsilon$ and thus prove continuity at $x$. □

□

## 8.3 Structural stability

We let $\mathcal{L}(\mathbb{R}^n)$ denote the space of all linear maps on $\mathbb{R}^n$. We have already seen two possible equivalence relations, linear conjugacy and topological conjugacy. We now define a topology on $\mathcal{L}(\mathbb{R}^n)$ and study the structurally stable maps and the bifurcations for both of these equivalence relations.

### 8.3.1 Structural stability for Two-dimensional maps

In the more general setting with $n \geq 2$ there is anyway a natural identification $\mathbb{L}(\mathbb{R}^n) \approx \mathbb{R}^{2n}$ given by the $2n$ entries of a matrix $A \in \mathbb{L}(\mathbb{R}^n)$. Thus, in this case also there is a natural norm induced by the Euclidean norm on $\mathbb{R}^n$. In the two dimensional case, given a matrix

$$B = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

we can therefore define a norm

$$\|A\| := \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2}$$

and from this, given another matrix $B$ we have

$$d(A, B) = \|A - B\| := \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2 + (a_4 - b_4)^2}$$

**Lemma 8.7.** *Every two-dimensional linear map is structurally unstable w.r.t. linear conjugacy.*

*Proof.* For any linear map $A$ and any $\epsilon > 0$ we can always find a "perturbation" $A_\epsilon$ with $d(A, A_\epsilon) < \epsilon$ such that $A$ and $A_\epsilon$ do not have the same eigenvalues, for example by changing just one of the entries of the matrix. Since $A, A_\epsilon$ do not have the same eigenvalues they cannot be linearly conjugate and thus do not belong to the same linear conjugacy class. Thus we have shown that $A$ does not belong to the interior of its conjugacy class. In fact, since $A$ is arbitrary this shows that none of the conjugacy classes have interiors. □

**Theorem 8.8.** *Let $A$ be a two-dimensional, invertible, hyperbolic linear map $A$ with distinct eigenvalues. Then $A$ is structurally stable w.r.t. topological conjugacy.*

*Proof.* Exercise. □

# Chapter 9

# Local linearization

Dynamical systems in higher dimensions can have extremely complex behaviour and we are still very far from any kind of classifications of all the possible dynamics. However in some cases we can describe the *local* dynamics in certain regions. First of all we need to define what we mean by a local topological conjugacy.

**Definition 9.1.** Let $X, Y$ be metric spaces, $f : X \to X$ and $g : Y \to Y$ continuous maps, and $p \in X$ and $g \in Y$ fixed points. We say that $f, g$ are *locally topologically conjugate* at $p$ and $q$ if there exists neighbourhoods $\mathcal{N}_p, \mathcal{N}_q$ of $p, q$ respectively, and a homeomorphism $h : \mathcal{N}_p \to \mathcal{N}_q$ such that $h \circ f = g \circ h$ *whenever both sides are defined.*

The definition does not assume that $\mathcal{N}_p$ and $\mathcal{N}_q$ are forward invariant by $f$ and $g$ respectively. Thus the conjugacy equation is not strictly well defined in all of $\mathcal{N}_p$. However, since $p, q$ are fixed, it is clearly defined in some neighbourhood of $p$ inside $\mathcal{N}_p$. Now let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$ diffeomorphism. Then, letting $x = (x_1, ..., x_n) \in \mathbb{R}^n$, the map $f$ is given by $n$ coordinate functions

$$f(x) = (f_1(x), ..., f_n(x))$$

and the derivative $Df(x)$ is a matrix

$$Df(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(x) & \cdots & \frac{\partial f_n}{\partial x_n}(x) \end{pmatrix}$$

The derivative matrix itself can therefore be considered as a *linear map*

$$Df(x) : \mathbb{R}^n \to \mathbb{R}^n.$$

A natural question is whether this linear map is somehow related or approximates the dynamics of the map $f$ around the point $x$. This question makes sense particularly if $x = p$ is a fixed point for $f$.

**Definition 9.2.** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$ map with a fixed point $p$. We say that $f$ is *locally linearizable* at $p$ if $f$ and $Df_p$ are *locally topologically conjugate* at $p$ and 0 respectively.

The fundamental result in this direction is the following. We first make the following definitions which generalizes naturally the definitions given above.

**Definition 9.3.** A linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ is hyperbolic if none of its eigenvalues lies on the unit circle. A fixed point $p$ if a $C^1$ diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ is *hyperbolic* if the derivative map $Df_p$ is hyperbolic.

**Theorem 9.4** (Grobman-Hartman Theorem, 1960)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$ map with a hyperbolic fixed point $p$. Then $f$ is locally linearizable at $p$.*

We shall not give a complete proof of this result but restrict our attention to the special case where all the eigenvalues of $Df_p$ are *inside* the unit circle, so that $Df_p$ is "contracting" in a sense to be made precise below. We first introduce the general notion of *contractions* and then prove a global version of thsi linearization in a very general infinite dimensional setting. Finally we explain how this implies the Grobman-Hartman Theorem for contracting fixed points.

## 9.1 The Contraction Mapping Theorem

Let $(X, d)$ be a metric space.

**Definition 9.5.** $f : X \to X$ is a *contraction* if there exists a constant $\lambda \in (0, 1)$ such that for all $x, y \in X$

$$d(f(x), f(y)) \leq \lambda d(x, y).$$

**Theorem 9.6.** *[Contraction Mapping Theorem] Suppose $X$ is a complete metric space and $f : X \to X$ is a contraction. Then there exists a unique, globally attracting, fixed point $p \in X$. Moreover, if $f_\omega : X \to X$ is a family of contractions with a uniform contraction constant $\lambda \in (0, 1)$ and depending continuously on the parameter $\omega$. Then the globally attracting fixed point $p$ depends continuously on the parameter $\omega$.*

To prove this result we first state and prove two simple lemmas.

**Lemma 9.7.** *Suppose $X$ is a metric space and $f : X \to X$ is a contraction. Then, for any $x \in X$ the sequence $\mathcal{O}^+(x) = \{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence.*

*Proof.* Recall that a sequence $\{x_n\}_{n=0}^{\infty}$ is Cauchy is for every $\epsilon > 0$ there exists $N_\epsilon$ such that $d(x_n, x_m) \leq \epsilon$ for every $n, m \geq N_\epsilon$. For the sequence $\{x_n\}_{n=0}^{\infty}$ given by iterates of $x = x_0$ we have, for every $n \geq 0$,

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq \lambda d(x_n, x_{n-1}) \leq \lambda^2 d(x_{n-1}, x_{n-2}) \leq \cdots \leq \lambda^n d(x_1, x_0).$$

Therefore, supposing without loss of generality that $n > m \geq N$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots d(x_{m+1}, x_m) \\ &\leq \lambda^{n-1} d(x_1, x_0) + \lambda^{n-2} d(x_1, x_0) + \cdots \lambda^m d(x_1, x_0) \\ &\leq d(x_1, x_0) \sum_{j=m}^{n-1} \lambda^j \leq d(x_1, x_0) \sum_{j=N}^{\infty} \lambda^j \end{aligned}$$

Since $\sum_{j=N}^{\infty} \lambda^j \to 0$ as $N \to \infty$, choosing $N = N_\epsilon$ sufficiently large we get $d(x_n, x_m) \leq \epsilon$ for all $n, m \geq N_\epsilon$. This shows that the sequence is Cauchy. □

By the definition of complete metric space, every Cauchy sequence converges. Thus, for every initial condition $x$ there exists a point $p \in X$ such that $x_n \to p$.

**Lemma 9.8.** *Let $X$ be a metric space and $f : X \to X$ continuous. Suppose that there exists $x \in X$ and $p \in X$ such that $x_n \to p$ as $n \to \infty$. Then $f(p) = p$.*

*Proof.* Let $y_n := f(x_n) = x_{n+1}$. Then clearly $y_n \to p$. By continuity of $f$ we have

$$f(p) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} y_n = p.$$

□

**Lemma 9.9.** *The fixed point given by the previous lemma does not depend on $x$, and thus is unique and globally attracting.*

*Proof.* If $p, \tilde{p}$ were distinct fixed points, we would have $d(f(p), f(\tilde{p})) = d(p, \tilde{p})$ which contradicts the assumption that $f$ is a contraction. Therefore the fixed point $p$ is unique and thus every orbit converges to $p$ and so $p$ is globally attracting. □

To complete the proof of the Theorem, for each parameter $\omega$, let $p_\omega = p(\omega)$ denote the unique globally attracting fixed point for $f_\omega$.

**Lemma 9.10.** *$p_\omega$ depends continuously on the parameter $\omega$.*

*Proof.* By the contraction and the triangle inequality, for any $x \in X$, we have

$$d(x, p_\omega) \leq d(x, f_\omega(x)) + d(f_\omega(x), p_\omega) \leq d(x, f_\omega(x)) + \lambda d(p_\omega, x)$$

which immediately gives

$$d(x, p_\omega) \leq \frac{1}{1-\lambda} d(x, f_\omega(x)).$$

In particular, choosing $x = p_{\tilde{\omega}}$ for some arbitrary $\tilde{\omega} \in \Omega$, we have

$$d(p_{\tilde{\omega}}, p_\omega) \leq \frac{1}{1-\lambda} d(p_{\tilde{\omega}}, f_\omega(p_{\tilde{\omega}}))$$

Since $f$ is continuous in $\omega$, then as $\omega \to \tilde{\omega}$ we have $f_\omega(p_{\tilde{\omega}}) \to f_{\tilde{\omega}}(p_{\tilde{\omega}}) = p_{\tilde{\omega}}$ and therefore $d(p_{\tilde{\omega}}, f_\omega(p_{\tilde{\omega}})) \to 0$ and $d(p_{\tilde{\omega}}, p_\omega) \to 0$. □

As a first application of the contraction mapping theorem, we prove the following.

**Proposition 9.11.** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$ diffeomoprhism and suppose $\|Df(x)\| < \lambda < 1$ for all $x \in \mathbb{R}^n$. Then there exists a unique globally attracting fixed point $p$.*

*Proof.* For simplicity let us consider the one-dimensional case $n = 1$ first. Then for any $x, y \in \mathbb{R}$, by the Mean Value Theorem, there exists $z$ such that $|f(x) - f(y)| = |f'(x)|\,|x - y| \leq \lambda|x - y|$. This implies that $f$ is a contraction on $\mathbb{R}$. In the general case, for any $x, y \in \mathbb{R}^n$, let $\tau = |x - y|$ and let $\gamma : [0, \tau] \to \mathbb{R}^n$ be the straight segment with $\gamma(0) = x, \gamma(\tau) = y$ and parametrized by arc length so that $|\gamma'(t)| = 1$ for all $t \in [0, \tau]$.Then we have

$$|f(x) - f(y)| \leq |f(\gamma)| = \int_0^\tau Df_{\gamma(t)}(\gamma'(t))dt \leq \int_0^\tau \|Df_{\gamma(t)}\|dt \leq \lambda\tau = \lambda|x - y|.$$

Again this implies that $f$ is a contraction on $\mathbb{R}^n$.
[1]

□

## 9.2 Perturbations of linear contractions

Any two contractions are to some extent "similar" since they both have a unique fixed point and all other orbits converge to this fixed point. However we cannot in

[1] there is an alternative proof: Consider the segment $\gamma(t) = (1-t)x + ty$, $t \in [0, 1]$ joining $x$ and $y$. Now, $f(\gamma(t))$ is a function of one variable and we can apply the mean value theorem.

$$\begin{aligned} |f(x) - f(y)| = |f(\gamma(0)) - f(\gamma(1))| &= |\frac{d}{dt} f(\gamma(t))|_{t=\tau}| \\ &= |Df(\gamma(\tau))(y - x)| \leq \|Df(\gamma(\tau))\||x - y| < \lambda|x - y|. \end{aligned}$$

general formalize this statement using the notion of conjugacy to show that any two contractions are topologically conjugate. We can do it in a special but very important case in which one of the contractions is a linear map and the other is a small perturbation of this linear map. We will state and prove this result in a very general setting of linear maps on (possibly infinite-dimensional) linear spaces. The proof will give a new strategy for constructing topological conjugacies.

Let $E$ be a Banach space with norm $\|\cdot\|$ and $T : E \to E$ an invertible linear map. We say that $T$ is a *linear contraction* if $\|T\| := \sup\{\|Tv\|/\|v\| : v \neq 0\} < 1$.

**Theorem 9.12.** *Let $E$ be a Banach space and $T : E \to E$ an invertible linear contraction. Suppose $f : E \to E$ is of the form $f = T + \Delta f$ with*

$$\Delta f \text{ bounded} \quad \text{and} \quad Lip(\Delta f) \leq \min\{\|T^{-1}\|^{-1}, 1 - \|T\|\}.$$

*Then $T$ and $f$ are topologically conjugate. More precisely, there exists a unique homeomorphism $h = Id_E + \Delta h$ with $\Delta h$ bounded and $h \circ T = f \circ h$.*

Before starting the proof of the Theorem we state and prove two propositions which will be applied in the proof.

**Proposition 9.13.** *Let $E$ be a Banach space and $T : E \to E$ an invertible linear contraction. Suppose $f : E \to E$ is of the form $f = T + \Delta f$ with $Lip(\Delta f) \leq \|T^{-1}\|^{-1}$. Then $f : E \to E$ is a homeomorphism.*

*Proof.* Since $f = T + \Delta f$ where $T$ is linear and $\Delta f$ is Lipschitz continuous, it follows immediately that $f$ is continuous. It only remains therefore to show that $f$ is invertible and that $f^{-1}$ is continuous. To show that $f$ is surjective we need to show that for every $y \in E$ there exists some $x \in E$ such that $f(x) = y$. By additionally showing that such an $x$ is unique we get that $f$ is also injective and thus invertible, and by showing that $x$ depends continuously on $y$ we will conclude that $x = f^{-1}(y)$ is continuous. For any $y \in E$ we have

$$f(x) = y \Leftrightarrow T(x) + \Delta f(x) = y \Leftrightarrow y - \Delta f(x) = T(x) \Leftrightarrow T^{-1}y - T^{-1} \circ \Delta f(x) = x$$

and so, defining the map $\theta_y : E \to E$ by

$$\theta_y(x) := T^{-1}y - T^{-1} \circ \Delta f(x),$$

the problem is reduced to showing $\theta_y$ has a unique fixed point and that this fixed point depends continuously on $y$. For $x, x' \in E$ we have

$$\begin{aligned}
\|\theta_y(x) - \theta_y(x)\| &= \|T^{-1} \circ \Delta f(x) + T^{-1} \circ \Delta f(x')\| \\
&= \|T^{-1} \circ (\Delta f(x) + \Delta f(x'))\| \\
&\leq \|T^{-1}\| \; \|\Delta f(x) + \Delta f(x')\| \\
&\leq \|T^{-1}\| Lip(\Delta f) \|x - x'\|.
\end{aligned}$$

By assumption $Lip(\Delta f) \leq \|T^{-1}\|^{-1}$ and therefore $\|T^{-1}\| Lip(\Delta f) < 1$ and so $\theta_y$ is contracting uniformly in $y$ and therefore has a unique fixed point which depends continuously on $y$ as required. $\square$

**Proposition 9.14.** *Let $E$ be a Banach space, $T : E \to E$ an invertible linear contraction. Suppose $f : E \to E$ is of the form $f = T + \Delta f$ with $Lip(\Delta f) \leq 1 - \|T\|$. Then $f$ has a unique fixed point.*

*Proof.* For any $x, y \in E$ we have

$$\begin{aligned}
\|f(x) - f(y)\| &\leq \|f(x) - T(x) + T(x) + T(y) - T(y) - f(y)\| \\
&\leq \|f(x) - T(x) - (f(y) - T(y)\| + \|T(x) - T(y)\| \\
&= \|(f - T)(x) - (f - T)(y)\| + \|T(x - y)\| \\
&\leq \|\Delta f(x) - \Delta f(y)\| + \|T\|\|x - y\| \\
&\leq Lip(\Delta f)\|x - y\| + \|T\|\|x - y\| \\
&= (Lip(\Delta f) + \|T\|)\|x - y\|
\end{aligned}$$

By assumption we have $Lip(\Delta f) + \|T\| < 1$ and therefore $f$ is a contraction and therefore has a unique fixed point. $\square$

*Proof of Theorem 9.12.* Let $g : E \to E$ be a map satisfying the same conditions as $f$, i.e. $g = T + \Delta g$ with $\Delta g$ bounded and $Lip(\Delta g) \leq \min\{\|T^{-1}\|^{-1}, 1 - \|T\|\}$. We will show that $f, g$ are topologically conjugate, more specifically there exists a unique homeomorphism $h$ of the form $h = Id_E + \Delta h$ with $\Delta h$ bounded such that

$$f \circ (Id_E + \Delta h) = (Id_E + \Delta h) \circ g. \tag{9.1}$$

This implies the result by taking $g = T$. Notice first of all that since $g$ is invertible by Lemma 9.13, equation (9.1) is equivalent to

$$(T + \Delta f) \circ (Id_E + \Delta h) \circ g^{-1} = Id_E + \Delta h$$

which in turn is equivalent to

$$T \circ \Delta h \circ g^{-1} + \Delta f \circ (id_E + \Delta h) \circ g^{-1} + T \circ g^{-1} - id_E = \Delta h \tag{9.2}$$

Thus it is sufficient to prove that there exists a unique bounded continuous function $\Delta h : E \to E$ such that (9.2) holds and such that $id_E + \Delta h$ is a homeomorphism. To prove the existence of $\Delta h$ satisfying (9.2), we formulate the question as a fixed point problem. Let $\mathcal{E} := C_b^0(\mathbb{E}, \mathbb{E})$ denote the space of bounded continuous maps on $E$. For $\varphi \in \mathcal{E}$ we let

$$\mathcal{F}(\varphi) := T \circ \varphi \circ g^{-1} + \Delta f \circ (id_E + \varphi) \circ g^{-1} + T \circ g^{-1} - id_E.$$

**Lemma 9.15.** *For every $\varphi \in \mathcal{E}$ we have $\mathcal{F}(\varphi) \in \mathcal{E}$.*

*Proof.* The first two terms are clearly bounded and because both $\varphi$ and $\Delta f$ are bounded. Thus we just need to show that $T \circ g^{-1} - id_E$ is bounded and this is true because $T \circ g^{-1} - Id_E = (T - g) \circ g^{-1}$ and $T - g = -\Delta g$ is bounded. Finally, $\mathcal{F}(\varphi)$ is just a sum and composition of continuous functions and thus is continuous. □

Lemma 9.15 implies that $\mathcal{F} : \mathcal{E} \to \mathcal{E}$ is a well defined map and that the existence of a bounded continuous function $\Delta h$ satisfying (9.2) is equivalent to the existence of a fixed point for $\mathcal{F}$ in $\mathcal{E}$. Notice moreover that $\mathcal{E}$ is a linear space and a Banach space with the supremum norm $\|\varphi\| := \sup_{x \in E} \|\varphi(x)\|$. We can therefore try to apply Proposition 9.14 to $\mathcal{F} : \mathcal{E} \to \mathcal{E}$. Indeed, we can write

$$\mathcal{F} = \mathcal{T} + \Delta\mathcal{F}$$

where

$$\mathcal{T}(\varphi) := T \circ \varphi \circ g^{-1}$$

and

$$\Delta\mathcal{F}(\varphi) := \Delta f \circ (Id_E + \varphi) \circ g^{-1} + T \circ g^{-1} - Id_E.$$

We then just need to verify that $\mathcal{T}, \Delta\mathcal{F}$ satisfy the assumptions of Proposition 9.14.

**Lemma 9.16.** *$\mathcal{T}$ is linear with $\|\mathcal{T}\|_{\mathcal{E}} \leq \|T\|_E < 1$.*

*Proof.* For $\varphi, \psi \in \mathcal{E}$ and $\alpha, \beta \in \mathbb{R}$, directly from the definition of $\mathcal{T}$ we have $\mathcal{T}(\alpha\varphi + \beta\psi) = T \circ (\alpha\varphi + \beta\psi) \circ g^{-1} = T \circ (\alpha\varphi \circ g^{-1} + \beta\psi \circ g^{-1}) = \alpha T \circ \varphi \circ g^{-1} + \beta T \circ \psi \circ g^{-1} = \alpha\mathcal{T}(\varphi) + \beta\mathcal{T}(\psi)$. Thus $\mathcal{T}$ is linear. Then

$$\|\mathcal{T}\|_{\mathcal{E}} = \sup_{\|\varphi\|_{\mathcal{E}}=1} \|\mathcal{T}(\varphi)\|_{\mathcal{E}} = \sup_{\|\varphi\|_{\mathcal{E}}=1} \|T \circ \varphi \circ g^{-1}\|_{\mathcal{E}}$$

From the definition of the norm $\| \cdot \|_{\mathcal{E}}$, for any $\varphi$ with $\|\varphi\|_{\mathcal{E}} = 1$ we have

$$\|T \circ \varphi \circ g^{-1}\|_{\mathcal{E}} \leq \|T\|_E \|\varphi \circ g^{-1}\|_{\mathcal{E}} \leq \|T\|_E < 1.$$

Therefore $\mathcal{T}$ is a linear contraction. □

**Lemma 9.17.** *$\Delta\mathcal{F}$ is Lipschitz with $Lip(\Delta\mathcal{F}) \leq Lip(\Delta f) \leq 1 - \|T\|$.*

Recall that by definition

$$Lip(\Delta\mathcal{F}) := \sup_{\varphi,\psi \in \mathcal{E}} \frac{\|\Delta\mathcal{F}(\varphi) - \Delta\mathcal{F}(\psi)\|_{\mathcal{E}}}{\|\varphi - \psi\|_{\mathcal{E}}} \text{ and } Lip(\Delta f) := \sup_{x,y \in E} \frac{\|\Delta f(x) - \Delta f(y)\|_E}{\|x - y\|_E}.$$

*Proof.* First of all that for any $\varphi, \psi \in \mathcal{E}$, by the definition of $\Delta\mathcal{F}$ we have

$$\begin{aligned}\|\Delta\mathcal{F}(\varphi) - \Delta\mathcal{F}(\psi)\|_{\mathcal{E}} &= \sup_{x\in E} \|\Delta f \circ (Id + \varphi) \circ g^{-1}(x) - \Delta f \circ (Id + \psi) \circ g^{-1}(x)\|_E \\ &= \sup_{y\in E} \|\Delta f \circ (Id + \varphi)(y) - \Delta f \circ (Id + \psi)(y)\|_E,\end{aligned}$$

where $y = g^{-1}(x)$, which is well defined. Since $\Delta f$ is Lipschitz we have

$$\|\Delta\mathcal{F}(\varphi) - \Delta\mathcal{F}(\psi)\|_{\mathcal{E}} \leq \text{Lip}(\Delta f)\|(Id+\varphi)(y) - (Id+\psi)(y)\|_E = \text{Lip}(\Delta f)\|\varphi - \psi\|_E. \tag{9.3}$$

This shows that $\text{Lip}(\Delta\mathcal{F}) \leq \text{Lip}(\Delta f)$.

□

We have shown therefore that $\mathcal{F} = \mathcal{T} + \Delta\mathcal{F}$ satisfies the assumptions of Proposition 9.14 and thus has a unique fixed point $\Delta h \in \mathcal{E}$. For such a function $\Delta h$ we have the following.

**Lemma 9.18.** *$Id + \Delta h : E \to E$ is a homeomorphism.*

*Proof.* Notice that by swapping $f, g$ in (9.1) and applying exactly the same arguments there exists another function $\overline{\Delta h} \in \mathcal{E}$ such that

$$g \circ (Id_E + \overline{\Delta h}) = (Id_E + \overline{\Delta h}) \circ f. \tag{9.4}$$

Composing both sides on the right by $Id + \Delta h$ and using (9.1) we have

$$g \circ (Id_E + \overline{\Delta h}) \circ (Id + \Delta h) = (Id_E + \overline{\Delta h}) \circ f \circ (Id + \Delta h) = (Id_E + \overline{\Delta h}) \circ (Id + \Delta h) \circ g$$

Similarly, using (9.1) and composing both sides of (9.4) on the left by $Id_E + \overline{\Delta h}$,

$$f \circ (Id + \Delta h) \circ (Id_E + \overline{\Delta h}) = (Id + \Delta h) \circ g \circ (Id_E + \overline{\Delta h}) = (Id + \Delta h) \circ (Id_E + \overline{\Delta h}) \circ f$$

Notice that for any two function $\varphi, \psi \in \mathcal{E}$ we have $(Id + \varphi) \circ (Id + \psi) = Id \circ (Id + \psi) + \varphi \circ (Id + \psi) = Id + \psi + \varphi + \varphi \circ \psi$ with $\psi + \varphi + \varphi \circ \psi \in \mathcal{E}$. Thus the equations above are of the same form as (9.1) and therefore, by the uniqueness of solutions we must have

$$(Id_E + \overline{\Delta h}) \circ (Id + \Delta h) = Id = (Id + \Delta h) \circ (Id_E + \overline{\Delta h}).$$

Thus $Id + \Delta h$ is a continuous bijection with continuous inverse $Id_E + \overline{\Delta h}$ and so is a homeomorphism. □

Combining the above results completes the proof. □

## 9.3 Adapted norms

The result above holds of course in particular for finite-dimensional contracting linear maps $A : \mathbb{R}^n \to \mathbb{R}^n$. Recall however, from the two-dimensional situation, that there are many linear maps which are not contracting but for which the conclusions of the contraction mapping theorem still holds, i.e. every initial condition converges to the unique fixed point at the origin. The perturbation theorem above cannot be immediately applied to these maps since they are not contractions, at least not in the given norm. However, using the fact that in many cases they are linearly conjugate to linear contractions (we proved above that this is the case for example for two-dimensional linear maps with distinct eigenvalues $\lambda_1, \lambda_2$ satisfying $\lambda := \min\{|\lambda_1|, |\lambda_2|\} < 1$, but it is in fact true in much greater generality) we can find equivalent *adapted* norms in which they are indeed contractions.

**Proposition 9.19.** *Suppose $A : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear map which is linearly conjugate to a linear map $B : \mathbb{R}^n \to \mathbb{R}^n$ with $\|B\| = \lambda < 1$. Then there exists a norm $\|\cdot\|_*$ such that $\|A\|_* < 1$.*

*Proof.* Let $P$ be the linear conjugacy between $A$ and $B$. Then, for every $n$ we have $A^n = PB^nP^{-1}$ and therefore $\|A^n\| = \|PB^nP^{-1}\| \leq \|P\|\|B^n\|\|P^{-1}\| \leq \|P\|\|P^{-1}\|\lambda^n$. Thus, taking $C = \|P\|\|P^{-1}\|$, for all $n \geq 1$ we have

$$\|A^n\| \leq C\lambda^n.$$

This shows that $A$ is "eventually contracting". Now choose $\lambda < \tilde{\lambda} < 1$ and $N \geq 1$ sufficiently large so that $C(\lambda/\tilde{\lambda})^{N-1} < 1$ and define a new norm by

$$\|v\|_* := \sum_{i=0}^{N-1} \tilde{\lambda}^{-i}\|A^i(v)\|$$

It is easy to check that this is indeed a norm. Then we have

$$\begin{aligned}
\|Av\|_* :&= \sum_{i=0}^{N-1} \tilde{\lambda}^{-i}\|A^i(Av)\| = \sum_{i=0}^{N-1} \tilde{\lambda}^{-i}\|A^{i+1}v\| = \tilde{\lambda}\sum_{i=0}^{N-1} \tilde{\lambda}^{-(i+1)}\|A^{i+1}v\| \\
&= \tilde{\lambda}\sum_{i=0}^{N-1} \tilde{\lambda}^{-i}\|A^iv\| - \tilde{\lambda}\|v\| + \tilde{\lambda}^{-(N-1)}\|A^Nv\| \\
&= \tilde{\lambda}\|v\|_* - \tilde{\lambda}\|v\| + \tilde{\lambda}^{-(N-1)}\|A^Nv\|
\end{aligned}$$

By the assumptions on $\tilde{\lambda}$ and $N$ we have

$$\tilde{\lambda}^{-(N-1)}\|A^Nv\| \leq C\tilde{\lambda}^{-(N-1)}\lambda^N\|v\| \leq C(\lambda/\tilde{\lambda})^{-N}\tilde{\lambda}\|v\| < \tilde{\lambda}\|v\|.$$

Therefore $\|Av\|_* < \tilde{\lambda}\|v\|_*$ for every $v$ and so $A$ is a contraction. □

## 9.4 Local linearization and structural stability

We are now ready to prove the Grobman-Hartman Theorem in the case for a hyperbolic fixed point with derivative $Df_p$ for which all eigenvalues are inside the unit circle. We have proved above (for two two dimensional case and with the additional assumption that the eigenvalues are distinct, but the result is true in general) that such a linear maps is linearly conjugate to a map in "canonical form" which is contracting, and therefore by Proposition 9.19 above, there is a norm for which $Df_p$ is contracting. We can therefore assume that we are in this norm and state the result as follows.

**Theorem 9.20.** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be $C^1$ with a fixed point $p$ with $Df_p$ contracting. Then $f$ is locally linearizable at $p$.*

This will follows easily from a local version of the conjugacy result given above.

**Proposition 9.21.** *Suppose $A : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $\|A\| < 1$. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is of the form $f = A + \Delta f$ where $\Delta f(0) = 0$ and there exists a neighbourhood $\mathcal{N}$ of 0 such that*

$$Lip(\Delta f|_{\mathcal{N}}) \leq \min\{\|A^{-1}\|^{-1}, 1 - \|A\|\}.$$

*Then $f$ and $A$ are locally topologically conjugate in a neighbourhood of 0.*

*Proof.* We can define a map $\hat{f} : \mathbb{R}^n \to \mathbb{R}^n$ such that $f|_{\mathcal{N}} = \hat{f}|_{\mathcal{N}}$ and such that $\hat{f} = A + \Delta\hat{f}$ with $\Delta\hat{f}$ bounded and $Lip(\Delta\hat{f}) \leq \min\{\|A^{-1}\|^{-1}, 1 - \|A\|\}$. Applying Theorem 9.12 we get a topological conjugacy between $\hat{f}$ and $A$. Restricting to $\mathcal{N}$ this gives a local topological conjugacy between $f$ and $A$ at the origin. □

*Proof of Theorem 9.20.* We can assume without loss of generality that the fixed point $p$ is at the origin, otherwise we can just define the map $\hat{f} = f - p$ which does indeed have a fixed point at the origin and satisfies the same conditions as $f$. Let $\Delta f := f - Df_0$. Since $f$ is $C^1$, $\Delta f$ is also $C^1$ and $D_0\Delta f = Df_0 - Df_0 = 0$ at 0. Therefore for any $\epsilon > 0$ there exists a neighbourhood $\mathcal{N}$ of the origin in which $\|D_x\Delta f\| < \epsilon$ and therefore $Lip(\Delta f|_{\mathcal{N}}) < \epsilon$. In particular, for $\epsilon > 0$ sufficiently small we have $Lip(\Delta f|_{\mathcal{N}}) \leq \min\{\|Df_0^{-1}\|^{-1}, 1 - \|Df_0\|\}$. We are therefore exactly in the setting of Proposition 9.21 which gives a local topological conjugacy between $f$ and $Df_0$ in a neighbourhood of 0. □

As a final corollary we get a local version of structural stability.

**Definition 9.22.** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$ map and $p$ a fixed point. We say that $f$ is $C^1$ *locally structurally stable* at $p$ if there exists $\epsilon > 0$ and a neighbourhood $\mathcal{N}$ of $p$ such that for any $g : \mathbb{R}^n \to \mathbb{R}^n$ with $d_1(f, g) < \epsilon$ there exists a fixed point $q$ for $g$ in $\mathcal{N}$ such that $f, g$ are locally topologically conjugate at $p, q$ respectively.

**Theorem 9.23.** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$ map and $p$ a fixed point with $Df_p$ contracting in some adapted norm. Suppose moreover that $Df_p$ is structurally stable in the space of linear maps. Then $f$ is locally structurally stable at $p$.*

*Remark* 9.24. We have shown above that a linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ is structurally stable in the space of linear maps if it is hyperbolic and has distinct eigenvalues $\lambda_1, \lambda_2$, thus the result applies in particular in this case. However it is in fact true more generally that any hyperbolic linear map in arbitrary dimension is structurally stable in the space of linear maps and thus we have preferred to state the result above in this more general setting.

*Proof.* Since $Df_p$ is contracting we have $\|Df_p\| < \lambda < 1$ in some adapted norm. Then there exists a neighbourhood $\mathcal{N}$ of $p$ such that $\|Df_x\| < \lambda < 1$ for all $x \in \mathcal{N}$ and for sufficiently small $\epsilon > 0$ we have $\tilde{\lambda} := \lambda + \epsilon < 1$ and $\|Dg_x\| < \tilde{\lambda} < 1$ for any $x \in \mathcal{N}$ and $g$ with $d_1(f, g) < \epsilon$. Thus $g$ is a contraction on $\mathcal{N}$ and has a unique fixed point $q \in \mathcal{N}$ which depends continuously on $g$. Thus Theorem 9.20 applies to both $f$ and $g$ which are locally topologically conjugate at $p, q$ respectively to their linear parts $Df_p, Dg_q$ respectively. Thus it is sufficient to show that $Df_p, Dg_q$ are topologically conjugate. But by assumption $Df_p$ is structurally stable in the space of linear maps and so, for $\epsilon$ sufficiently small, $Df_p, Dg_q$ are topologically conjugate. □

# Part IV

# Chaotic Dynamics

# Chapter 10

# Symbolic Dynamics

So far we have studied systems which have a unique or a finite number of fixed points and for which the dynamics of essentially all remaining points is associated in some way to these fixed points. In this part of the notes we introduce systems which have a much richer orbit structure. This necessitates the introduction of a very important technique called *symbolic coding*, in order to study the problems of topological conjugacy and classification.

## 10.1 Symbolic coding

Let $X$ be a set, $\mathcal{P} = \{I_0, ..., I_{s-1}\}$ a finite partition of $X$, and $f : X \to X$ a map. Then any initial condition $x_0 \in X$ belongs to some element of the partition $\mathcal{P}$ and also every iterate $x_n = f^n(x_0)$ belongs to some element of $\mathcal{P}$. Thus we can associate to $x_0$ an infinite sequence

$$x_0 \mapsto \underline{a} = a_0 a_1 a_2$$

where

$$a_i = \ell \quad \text{if} \quad x_i = f^i(x) \in I_\ell$$

with $\ell \in \{0, ..., s-1\}$. This "coding" of the orbit is naturally related to the dynamics in the sense that the sequence associated to every forward iterate of $x_0$ is automatically contained in the sequence associated to $x_0$. Indeed, it is clear that if $a_0 a_1 a_2 ...$ is the sequence associated to $x_0$ then the sequence associated to $x_1 = f(x_0)$ is $a_1 a_2 a_3 ....$ and more generally, the sequence associated to $x_k$ for $k \geq 0$ is the sequence $a_k a_{k+1} a_{k+2} .....$ To formalize this idea it is useful to introduce the *space of sequences* or *symbolic space*

$$\Sigma_s^+ := \{\underline{a} = a_0 a_1 a_2 ..., \quad a_i \in \{0, .., s-1\}\}.$$

This is a set whose elements are sequences and we can define a natural map $\sigma : \Sigma_s^+ \to \Sigma_s^+$ as follows: for any $\underline{a} \in \Sigma$ we define

$$\sigma(a_0 a_1 a_2 a_3 ....) = a_1 a_2 a_3 ....$$

Thus the map $\sigma$ drops the first element of the sequence. For reasons which will become clear later it is useful to think that it "shifts" the sequence to the left, and for this reason it is sometimes called the *shift map*. The coding mentioned above can be thought of as a map

$$\tilde{\pi} : X \to \Sigma_s^+$$

where

$$\tilde{\pi}(x_0) = a_0 a_1 a_2 ... \quad a_i = \ell \quad \text{if} \quad x_i = f^i(x) \in I_\ell.$$

Then it is easy to see that

$$\tilde{\pi}(f(x_0)) = \tilde{\pi}(x_1) = a_1 a_2 a_3 .... = \sigma(a_0 a_1 a_2 ...) = \sigma \tilde{\pi}(x_0)$$

Therefore we conclude that

$$\tilde{\pi} \circ f = \sigma \circ \tilde{\pi}. \tag{10.1}$$

Notice that in general $\tilde{\pi}$ is not a bijection and therefore $\tilde{\pi}$ is not necessarily a conjugacy. Still, we call any function satisfying (10.1), a *semiconjugacy*.

The question on whether this construction can yield any useful information depends essentially on two things: i) is $\tilde{\pi}$ a bijection? In which case certain features of the map $f$ will be reflected in features of the map $\sigma$, and ii) can we understand the dynamics of the map $\sigma$? In this chapter we will focus on the abstract setting of the shift map and its dynamical properties, then consider some systems for which we can show that $\tilde{h}$ is a conjugacy. First we give some very simple examples of the coding procedure to get an initial feeling for what the procedure can give.

*Example* 21. A most trivial example is if the partition $\mathcal{P}$ is trivial and is made up of a single elements corresponding to the whole space $X$. In that case the symbolic space $\Sigma_1^+$ is trivial, corresponds to a single point, and even though the conjugacy still holds, it does not provide any useful information.

*Example* 22. Another useful example is given by a contraction with a unique fixed point. If one of the partition elements, say $I_t$, contains a neighbourhood of the fixed point, then after some every initial condition $x$ will fall into this neighbourhood and never leave. This means that the symbolic sequence of every point will terminate with an infinite number of consecutive occurrences of the symbol $t$. Once again the semi-conjugacy holds but in some sense is not very useful. There will be only a countable number of distinct sequences and this there will be infinitely many points which map to the same sequence, i.e. $\tilde{\pi}$ will be infinite to 1.

## 10.2 The symbolic space

We begin by taking a closer look at the symbolic space. To study additional properties of the dynamics we introduce a metric on $\Sigma_s^+$, for $\underline{a}, \underline{b} \in \Sigma_s^+$ we let

$$d(\underline{a}, \underline{b}) := \sum_{i=0}^{\infty} \frac{|a_i - b_i|}{2^i}. \tag{10.2}$$

**Lemma 10.1.** *The function $d(\cdot,\cdot)$ is a metric on $\Sigma_s^+$.*

*Proof.* Exercise 10.4.1. □

*Remark* 10.2. The topology induced by this metric is the so-called *product topology* given by considering the discrete topology on the finite set $\{0, ..., s-1\}$ and noting that $\Sigma_s^+$ can be written as an Cartesian product of this set as $\Sigma_s^+ = \{0, ..., s-1\}^{\mathbb{N}}$. There are other equivalent metrics which are sometimes used in the literature and which induce the same topology: for example we can define $\tilde{d}(\underline{a}, \underline{b})$ by $2^{-\kappa}$ where $\kappa$ is the largest integer such that $a_i = b_i$ for all $i \leq \kappa$, see Exercise 10.4.2.

A key property of any metric which induces the product topology is that two sequences are close if and only their terms coincide for a sufficiently large initial block. We will use this property repeatedly below and therefore, for convenience, formalize it as follows.

**Lemma 10.3.** *For every $\epsilon > 0$ there exists $n_\epsilon > 0$, with $n_\epsilon \to \infty$ when $\epsilon \to 0$, such that if $\underline{a}, \underline{b} \in \Sigma_2^+$ satisfy $a_i = b_i$ for all $i = 0, ..., n_\epsilon$ then $d(\underline{a}, \underline{b}) < \epsilon$. Conversely, for every $n > 0$ there exists $\epsilon_n > 0$, with $\epsilon_n \to 0$ when $n \to \infty$, such that if $d(\underline{a}, \underline{b}) < \epsilon_n$ then $a_i = b_i$ for all $i = 0, ..., n$.*

*Proof.* Exercise 10.4.3. □

Having a metric space structure allows us to describe the dynamics of both the space $\Sigma_2^+$ and the map $\sigma$ in much more detail. First of all we have

**Lemma 10.4.** *$\Sigma_s^+$ is a Cantor set*

*Proof.* A Cantor set is by definition a set which is compact, perfect, totally disconnected and uncountable). Compactness follows by the observation that $\Sigma_s^+$ is a product of compact sets and therefore, by Tychonoff's theorem is itself compact. Uncountability follows by the standard Cantor's diagonal argument. Perfect means there are no isolated points and totally disconnected means that connected components are points. We leave these two properties as exercise 10.4.4. □

**Lemma 10.5.** *The shift map $\sigma : \Sigma_2^+ \to \Sigma_2^+$ is continuous.*

*Proof.* Exercise 10.4.5. □

All the definitions and properties mentioned above can easily be extended to the space
$$\Sigma_s := \{0, ..., s-1\}^{\mathbb{Z}} = \{.....a_{-2}a_{-1}a_0a_1a_2....\}$$
of *bi-infinite sequences* equipped with the metric

$$d(\underline{a}, \underline{b}) := \sum_{i=-\infty}^{\infty} \frac{|a_i - b_i|}{2^{|i|}} \tag{10.3}$$

and the shift map defined by

$$\sigma(....a_{-2}a_{-1}a_0a_1a_2....) = (....b_{-2}b_{-1}b_0b_1b_2....) \text{ where } b_i = a_{i+1} \ \forall\ i \in \mathbb{Z}.$$

## 10.3 The shift map

It is now fairly straightforward to prove two basic properties of the dynamics of the shift map. It is quite easy to see that both in the invertible and non-invertible cases

**Lemma 10.6.** *The set $Per(\sigma)$ of periodic points of $\sigma$ is dense in $\Sigma_s^+$.*

*Proof.* Exercise 10.4.6. □

**Lemma 10.7.** *The map $Per(\sigma)$ is transitive on $\Sigma_s^+$.*

*Proof.* Exercise 10.4.7. □

## 10.4 Exercises

**Exercise 10.4.1.** Show that

$$d(\underline{a}, \underline{b}) := \sum_{i=0}^{\infty} \frac{|a_i - b_i|}{2^i}.$$

is a metric[1] on $\Sigma_s^+$.

[1]Recall that for an arbitrary set $X$, $d(x, y)$ is a *metric* on $X$ if it satisfies the following conditions: i) $d(x, y) \geq 0\ \forall\ x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$; ii) $d(x, y) = d(y, x)$ for all $x, y \in X$; iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

**Exercise 10.4.2.** Show that

$$\tilde{d}(\underline{a},\underline{b}) := 2^{-\kappa} \text{ where } \kappa \text{ is the largest integer such that } a_i = b_i \text{ for all } i \leq \kappa$$

is also a metric on $\Sigma_s^+$. Show that $d, \tilde{d}$ are equivalent.

**Exercise 10.4.3.** Show that for every $\epsilon > 0$ there exists $n_\epsilon > 0$, with $n_\epsilon \to \infty$ when $\epsilon \to 0$, such that if $\underline{a}, \underline{b} \in \Sigma_2^+$ satisfy $a_i = b_i$ for all $i = 0, ..., n_\epsilon$ then $d(\underline{a},\underline{b}) < \epsilon$. Show, conversely, for every $n > 0$ there exists $\epsilon_n > 0$, with $\epsilon_n \to 0$ when $n \to \infty$, such that if $d(\underline{a},\underline{b}) < \epsilon_n$ then $a_i = b_i$ for all $i = 0, ..., n$.

**Exercise 10.4.4.** Show that $\Sigma_s^+$ is perfect and totally disconnected.

[*A topological space $X$ is perfect if it has no isolated points, and totally disconnected if its connected components are single points, i.e. for any $x, y \in X$ there exist disjoint open sets $A, B$ with $x \in A, y \in B, A \cup B = X$.* ]

**Exercise 10.4.5.** Show that the shift map $\sigma : \Sigma_s^+ \to \Sigma_s^+$ is continuous.

**Exercise 10.4.6.** Show that the set $Per(\sigma)$ of periodic points of $\sigma$ is dense in $\Sigma_s^+$.

**Exercise 10.4.7.** Show that $\sigma$ is transitive on $\Sigma_s^+$.

# Chapter 11

# Dynamically defined Cantor sets

## 11.1 The Tent map family

We will apply the coding method to a relatively simple class of maps. Consider the family of maps $f_\lambda : \mathbb{R} \to \mathbb{R}$, for $\lambda > 0$, defined by

$$f_\lambda(x) = \begin{cases} \lambda x & \text{if } x \le 0.5 \\ -\lambda x + \lambda & \text{if } x \ge 0.5 \end{cases} \tag{11.1}$$

This is sometimes called the family of *tent maps*. The dynamical properties of tent maps depends on the parameter $\lambda$. It is easy to see that the orbit of every $x \notin [0,1]$ satisfies $f^n(x) \to -\infty$ as $n \to \infty$. Letting $I := [0,1]$ we write

$$I = I_0 \cup \Delta \cup I_1$$

where

$$I_0 = [0, 1/\lambda], \quad \Delta = (1/\lambda, (\lambda-1)/\lambda), \quad I_1 = [(\lambda-1)/\lambda, 1].$$

Now let

$$\Lambda = \{x : f^n(x) \in I_0 \cup I_1 \text{ for all } n \ge 0\}.$$

A priori we do not know whether $\Lambda$ contains any other points besides the fixed points but by definition it is invariant in the sense that $f(\Lambda) \subseteq \Lambda$ and therefore we can define the map $f$ restricted to $\Lambda$. We will denote this by $f|_\Lambda$.

**Theorem 11.1.** *$\Lambda$ is a Cantor set, $Per(f)$ is dense in $\Lambda$, and $f|_\Lambda$ is transitive.*

We will prove this Theorem by using the technique of symbolic coding and showing that $f|_\Lambda$ is topologically conjugate to $\sigma|_{\Sigma_2^+}$.

Notice first of all that by definition $\Lambda \subseteq I_0 \cup I_1$ and therefore, since $I_0 \cap I_1 = \emptyset$, the restrictions $\Lambda_0 := I_0 \cap \Lambda$ and $\Lambda_1 := I_1 \cap \Lambda$ form a partition of $\Lambda$. A point $x \in \Lambda$

satisfies $x \in \Lambda_0 \Leftrightarrow x \in I_0$ and $x \in \Lambda_1 \Leftrightarrow x \in I_1$. Thus we can define the symbolic coding of every point $x \in \Lambda$ by

$$\pi(x) = \underline{a}(x) := a_0 a_1 a_2 a_3 ... \text{ where } a_i \in \{0, 1\} \text{ and } f^i(x) \in I_{a_i} \quad \forall\, i \in \mathbb{N}.$$

Thus the sequence $\underline{a}$ describes the "combinatorics" of the forward orbit of the point $x$ in terms of the two intervals $I_0, I_1$. We need to show that $\tilde{\pi}$ is a bijection. For any $\underline{a} = (a_0 a_1 a_2 ...) \in \Sigma_2^+$ and any $n \geq 0$ let Let

$$I_{a_0 a_1 \ldots a_n} := \{x \in I : f^i(x) \in I_{a_i}, i = 0, \ldots, n\}.$$

denote the set of points who share the same combinatorics, as defined by the initial terms of $\underline{a}$, up to time $n$.

**Lemma 11.2.** *For all $\underline{a} \in \Sigma_2^+$ and $n \geq 0$, $I_{a_0 a_1 \ldots a_n}$ is a non-empty closed interval.*

*Proof.* We will show that for each $n \geq 0$ there exist exactly $2^{n+1}$ pairwise disjoint closed intervals corresponding exactly to all possible finite sequences $a_0 a_1 ... a_n$. We will prove this by induction.

*First step of the induction:* We have the two disjoint closed intervals $I_0, I_1$ and the maps $f : I_0 \to I$ and $f : I_1 \to I$ are bijections.

*Inductive assumption:* Suppose that for some $n \geq 0$ there exist $2^{n+1}$ disjoint non-empty closed intervals $I_{a_0 \ldots a_n}$ corresponding to all possible finite sequences $a_0 ... a_n$, such that for each finite sequence $a_0 ... a_n$ and that the map

$$f^n : I_{a_0 \ldots a_n} \to I \quad \text{ is a bijection .} \tag{11.2}$$

*General step of the induction:* By (11.2), the interval $I_{a_0 \ldots a_n}$ contains two disjoint closed subintervals $I_{a_0 \ldots a_{n-1} 0}$ and $I_{a_0 \ldots a_{n-1} 1}$ such that

$$f^n : I_{a_0 \ldots a_{n-1} 0} \to I_0 \quad \text{ and } \quad f^n : I_{a_0 \ldots a_{n-1} 1} \to I_1$$

are bijections and therefore, since $f : I_0 \to I$ and $f : I_1 \to I$ are also bijections, their compositions

$$f^{n+1} : I_{a_0 \ldots a_{n-1} 0} \to I \quad \text{ and } \quad f^{n+1} : I_{a_0 \ldots a_{n-1} 1} \to I$$

are bijections. $\square$

Now let

$$I_{\underline{a}} := \bigcap_{n \in \mathbb{N}} I_{a_0 a_1 \ldots a_n} = \{x \in I : f^i(x) \in I_{a_i}, i \in \mathbb{N}\}$$

denote the set of points whose orbits have exactly the combinatorics described by the full sequence $\underline{a}$.

**Corollary 11.3.** *For all $\underline{a} \in \Sigma_2^+$, $I_{\underline{a}} \neq \emptyset$.*

*Proof.* By definition $I_{a_0a_1\ldots a_{n-1}a_n} \subseteq I_{a_0a_1\ldots a_{n-1}} \subseteq \cdots \subseteq I_{a_0} \subseteq I$ where, by Lemma 11.2, each interval in the nested sequence is a non-epty closed interval. The statement then follows from the general topological result that the countable intersection of non-empty nested closed sets is always non-empty. $\square$

**Lemma 11.4.** *For all $\underline{a} \in \Sigma_2^+$, $I_{\underline{a}}$ is a single point.*

*Remark* 11.5. Notice that it does not follow automatically from the nested property of the intervals $I_{a_0\ldots a_n}$ that their intersection is a single point, see Exercise **??**.

*Proof.* By the inductive construction above we have that for each $n$, the map (11.2) is affine and $|(f^n)'(x)| = \lambda^n$ for all $x \in I_{a_0\ldots a_n}$ and so, by the Mean Value Theorem, we have $|I_{a_0\ldots a_n}| = \lambda^{-(n+1)}$. Thus $|I_{a_0\ldots a_n}| \to 0$ as $n \to \infty$ and so $|I_{\underline{a}}| = 0$ and so $I_{\underline{a}}$ is a single point. $\square$

*Remark* 11.6. For future reference we observe that the map (11.2) also implies that the two "new intervals" $I_{a_0\ldots a_{n-1}0}$ and $I_{a_0\ldots a_{n-1}1}$, each of whose length is $\lambda^{-(n+1)}$, are separated by a a component of the pre image of $\Delta$ which also has length exactly $(1-2\lambda^{-1})^{-(n+1)}$. In particular we have that *two points $x, y \in \Lambda$ with $|x-y| < 3^{-(n+1)}$ necessarily belong to the same interval of the form $I_{a_0\ldots a_n}$. Conversely, if two points $x, y \in \Lambda$ satisfy $|x-y| > (1-2\lambda^{-1})^{-(n+1)}$ then they necessarily belong to two distinct intervals of the form $I_{a_0\ldots a_n}$.*

**Proposition 11.7.** *The map $\pi : \Lambda \to \Sigma_2^+$ is a bijection.*

*Proof.* Exercise **??**. $\square$

Proposition 11.7 immediately implies the followinfg

**Lemma 11.8.** *The maps $\sigma$ on $\Sigma_2^+$ and $f$ on $\Lambda$ are conjugate. Threfeore $f|_\Lambda$ has exactly $2^n$ periodic points of period $n$ for every $n \geq 1$ and in particular it has an infinite number of periodic points.*

*Proof.* Follows by theProposition 11.7 and the properties of the dynamics of $\sigma$. $\square$

**Proposition 11.9.** *$f|_\Lambda$ is topologically conjugate to $\sigma|_{\Sigma_2^+}$.*

*Proof.* Exercise **??**. $\square$

**Corollary 11.10.** *$\Lambda$ is a Cantor set, the set $Per(f|_\Lambda)$ of periodic points of $f|_\Lambda$ is dense in $\Lambda$, and $f|_\Lambda$ is transitive.*

*Proof.* Follows by the topological conjugacy and the properties of $\sigma$. $\square$

## 11.2 Nonlinear maps

The construction above can be easily extended to much more general situations. Let $f : I \to \mathbb{R}$ and $I_0, ..., I_{s-1}$ be closed disjoint subintervals of $I$ and suppose there exists $\lambda > 1$ such that for each $i = 0, .., s-1$:

1) $f : I_i \to I$ is a $C^1$ diffeomorphism;

2) $|f'(x)| \geq \lambda > 1$ for all $x \in I_i$.

Then we define the "maximal invariant" set in the union of intervals $I_i$ by

$$\Lambda := \{x : f^n(x) \in I_0 \cup \cdots I_{s-1} \text{ for all } n \geq 0\}.$$

By definition $\Lambda$ is a forward invariant set and we can define $f|_\Lambda$.

**Theorem 11.11.** *$f|_\Lambda$ is topologically conjugate to $\sigma|_{\Sigma_s^+}$. In particular $\Lambda$ is a Cantor set, $Per(f)$ is dense in $\Lambda$ and $f|_\Lambda$ is transitive.*

*Proof.* Exercise **??**. □

*Remark* 11.12. We assume for simplicity that $f$ is defined on all of $I$ but this is not necessary as we only ever consider the dynamics in $I_0 \cup \cdots \cup I_{s-1}$.

*Remark* 11.13. Notice that the specific way in which the symbolic space is "embedded" in the interval $I$ depends on the sign of the derivative on each branch.

*Remark* 11.14. Notice that the assumption 2) in the above results can be further weaked as follows:

2') that there exists constants $C, \lambda > 1$ such that for every $x$ and every $n \geq 1$,

$$x, f(x), f^2(x), ..., f^{n-1}(x) \in I_0 \cup \cdots \cup I_{s-1} \quad \Longrightarrow \quad |(f^n)'(x)| > C\lambda^n. \tag{11.3}$$

Symbolic dynamics this gives a dynamical model of the dynamics of some quite complicated invariant sets. However perhaps the most interesting application of this result is to show that two such maps are topologically conjugate to each other. Indeed, suppose that $f : I \to I$ is as above and $g : I \to \mathbb{R}$ is another map satisfying similar conditions: there exist closed disjoint subintervals $I'_0, ..., I'_{\ell-1}$ be closed disjoint subintervals of $I$ and $\lambda' > 1$ such that for each $i = 0, .., \ell - 1$:

1) $g : I_i \to I$ is a $C^1$ diffeomorphism;

2) $|g'(x)| \geq \lambda$ for all $x \in I'_i$.

Then we let

$$\Lambda' := \left\{x : g^n(x) \in I'_0 \cup \cdots I'_{\ell-1} \text{ for all } n \geq 0\right\}.$$

**Corollary 11.15.** *$f|_\Lambda$ and $g|_{\Lambda'}$ are topologically conjugate.*

## 11.3 Sensitive dependence on initial conditions

To conclude this chapter we also prove that both $f|_\Lambda$ and $\sigma|_{\Sigma_2^+}$ satisfy another property which is more metric than topological and has been one of the first properties to be used in attempts to formalise the notion of *chaotic* dynamical systems.

**Definition 11.16.** Let $X$ be a metric space and $f : X \to X$ a map. We say that $f$ exhibits *sensitive dependence on initial conditions* if there exist $\epsilon > 0$ such that for all initial conditions $x_0$ and every $\delta > 0$ there exists $y_0 \in X$ with $d(x_0, y_0) < \delta$ and $n > 0$ such that $d(x_n, y_n) \geq \epsilon$.

**Lemma 11.17.** *The shift map $\sigma|_{\Sigma_s^+}$ and the conjugate map $f|_\Lambda$ both exhibit sensitive dependence on initial conditions.*

*Proof.* □

## 11.4 Exercises

**Exercise 11.4.1.** Let $\{J_n\}$ be a collection of nested intervals of the form $J_n = [a_n, b_n]$. Show that $J = \bigcap_{n=0}^{\infty} J_n = [a, b]$ is nonempty and connected.

**Exercise 11.4.2.** Find a sequence $J_n$ of nested intervals such that $\bigcap_n J_n = \emptyset$.

**Exercise 11.4.3.** Find a sequence $J_n$ of closed nested intervals such that $\bigcap_n J_n$ is a non-trivial interval.

**Exercise 11.4.4.** Let $f : I \to \mathbb{R}$ and $I_0, ..., I_{s-1}$ be closed disjoint subintervals of $I$ and suppose there exists $\lambda > 1$ such that for each $i = 0, .., s-1$:

1. $f : I_i \to I$ is a $C^1$ diffeomorphism;
2. $|f'(x)| \geq \lambda > 1$ for all $x \in I_i$.

Let

$$\Lambda := \{x : f^n(x) \in I_0 \cup \cdots I_{s-1} \text{ for all } n \geq 0\}$$

and $\pi : \Lambda \to \Sigma_s^+$ be given by

$$\pi(x) = \underline{a}(x) := a_0 a_1 a_2 a_3 ... \text{ where } a_i \in \{0, ...s-1\} \text{ and } f^i(x) \in I_{a_i} \quad \forall\, i \in \mathbb{N}.$$

Show that

1. $\pi$ is a bijection;
2. $\pi$ is continuous;
3. $\pi^{-1}$ is continuous.

Conclude that $f|_\Lambda$ is topologically conjugate to $\sigma|_{\Sigma_s^+}$. *Hint: to prove continuity, you can use the* $\epsilon - \delta$ *definition and the following observations: i) Let* $\eta$ *denote the minimum distance between any two intervals* $I_i, I_j$ *for* $i, j \in \{0, ..., s-1\}$*. Then if* $x, y \in \Lambda$ *with* $d(x, y) < \eta$ *it follows that* $x, y$ *belong to the same interval* $I_i$*. ii) Let* $\lambda_{max} = \max_{x \in I_0 \cup \dots \cup I_{s-1}} \{|f'(x)|\}$*, then for any two points* $x, y \in \Lambda$*,* $|f^n(x), f^n(y)| \leq \lambda_{max}^n |x - y|$*. iii) Exercise (3) in Problem Sheet 4A.*

# Chapter 12

# Full branch maps

For the family of tent maps, defined above, for $\lambda > 2$ we get the dynamically defined Cantor sets from the previous section. However for $\lambda = 2$ the situation is a bit different. Even though we still have two closed intervals $I_0, I_1$ such that $f_\lambda : I_0 \to I$ and $f_\lambda : I_1 \to I$ are $C^1$ diffeomorphisms (indeed, affine), these intervals are no longer disjoint. This has an effect on the coding procedure because the coding is no longer uniquely defined for any orbit which falls in the intersection $I_0 \cap I_1$. Fortunately, this intersection is very small (a single point) and so it turns out that the problem can be overcome. Rather than studying this particular case in detail, we formulate the general class of maps of this kind which we will study.

**Definition 12.1.** $f : I \to I$ is a *full branch piecewise expanding* if there exist disjoint *open* intervals $I_0, ..., I_{\ell-1}$ and a constant $\lambda > 1$ s.t. for $i = 0, .., \ell - 1$:
1) $\overline{f(I_i)} = I$.
2) $|f'(x)| \geq \lambda$ for all $x \in I_i$.
3) $I = \overline{I_0 \cup \cdots \cup I_{\ell-1}}$

If a full branch piecewise expanding map has $\ell$ intervals satisfying the definition then we say that $f$ has $\ell$ branches. Notice that the key difference between these maps and those studied in the previous chapter is contained in item 3) which implies that there are no *gaps*. In this case we can in principle still define the set $\Lambda$ as the set of points which remain in $I$ for all $n \geq 0$, but since we now have $f(I) = I$ we have $\Lambda = I$.

## 12.1 Expansions of real numbers

A particularly interesting example of a class of full branch piecewise expsnding maps are maps of the form $f(x) = \lambda x \mod 1$ for $\lambda \in \mathbb{N}$, $\lambda \geq 2$. Notice that the map is still perfectly well defined even if $\lambda > 0$ is not an integer but in that case if is not full branch. If $\lambda$ is an integer then it is full branch and it has exactly $\ell = \lambda$ branches. In these examples the symbolic coding of points coincides exactly with the so-called *base $\lambda$ expansion*. Consider for example the special case $\lambda = 10$. Then the map $f$ is given by

$$f(x) = 10x \mod 1.$$

Using the formalism introduced above, $f$ is a piecewise expanding map with the branches defined on the collection of open intervals

$$I_0 = (0, 0.1), I_1 = (0.1, 0.2), ........., I_8 = (0.8, 0.9), I_9 = (0.9, 1).$$

Then, if we were to construct the symbolic coding of any point whose forward orbit remained forever in the union of these open intervals we would get exactly the decimal expansion of the initial condition. This leaves open the question of how to deal with the boundary points, and a related issue is about deciding the action of the map on the boundary points. We have two options:

*Option 1:* We could take all subintervals half-open and half-closed:

$$I_0 = [0, 0.1), I_1 = [0.1, 0.2), ........., I_8 = [0.8, 0.9), I_9 = [0.9, 1)$$

and define $f(0.1) = f(0.2) = \cdots = f(0.9) = 0$. This would then allow us to include these boundary points as points with a well defined combinatorics. We could even include the point 1 simply be defining $f(1) = 1$ and adjoing it to the extreme interval which we could take closed:

$$I_0 = [0, 0.1), I_1 = [0.1, 0.2), ........., I_8 = [0.8, 0.9), I_9 = [0.9, 1].$$

We would then have a partition of the interval and a well defined unique symbolic coding for each point. The "problem" here of course is that not every symbol appears, so that map $\pi : \Lambda \to \Sigma_{10}^+$ which associates to each point its symbolic coding is not surjective. In fact, except for the symbolic coding of the fixed point 1, which would be the sequence 99999...., no other point would have a sequence ending in 9's. If we defined these as decimal expansions of points then they would be uniquely defined but not all sequences would correspond to a decimal expansion.

*Option 2:* If we try to code using the closure of the intervals:

$$I_0 = [0, 0.1[, I_1 = [0.1, 0.2], ........., I_8 = [0.8, 0.9], I_9 = [0.9, 1].$$

then we have the problem of deciding the value of the map on the boundaries: is $f(0.1) = 0$ or $f(0.1) = 1$? One way to resolve this problem is to identify 0 and 1, which is fact natural in the setting of studying a map where we take points "mod 1". In terms of the partition into closed intervals, the symbolic codings of

0 and 1 would be 0000... and 9999.... respectively. Thus identifying these two points, means that there is an ambiguity in the its symbolic coding. Modulo this ambiguity we no longer have a problem about the image of the end points since they map a well defined unique point. However of course these boundary points now also have two possible symbolic codings: for example the coding of the point 0.4 can be either 39999.... or 400000..... The same is true of course for all preimages of the boundary points. For example if $f^k(x) = 0.4$, then the coding of $x$ will be either $a_0a_1, .., a_{k-1}399999$ or $a_0a_1, .., a_{k-1}400000$..... Using this approach it is then more natural to define the map

$$\pi : \Sigma_{10}^{+} \to \Lambda$$

which assigns to each sequence the corresponding point $\pi(\underline{a}) = I_{\underline{a}}$, which is therefore not injective at all preimages of $0 = 1$ mod1.

The same discussion can be carried out for all other expansions. Indeed, recall that for an integer $\lambda \geq 2$, the base $\lambda$ expansion of a number is given by the sequence $x_0x_1x_2...$ so that

$$x = \frac{x_0}{\lambda} + \frac{x_1}{\lambda^2} + \frac{x_2}{\lambda^3} + \dots$$

It is easy to see then that the map $f(x) = \lambda x \bmod 1$ corresponds exactly to the action on the shift on the expansion. Moreover such a map is a piecewise expanding map with $\lambda$ branches and the lack of uniqueness for the $\lambda$ expansions can be explained dynamically exactly as in the case of $\lambda = 10$.

*Remark* 12.2. We remark finally that there is also another way to see this lack of uniqueness of expansions. Indeed, for $\underline{a}, \underline{b} \in \Sigma_\ell^+$, let

$$d_E(\underline{a}, \underline{b}) = \left| \sum_{i=0}^{\infty} \frac{a_i - b_i}{\ell^i} \right|,$$

Then is is easy to check that $d_E$ is a *pseudo-metric*, i.e. satisfies $d_E(\underline{a}, \underline{b})$ and the triangle inequality but not the property of a metric that says that $d_E(\underline{a}, \underline{b}) = 0$ if and only if $\underline{a} = \underline{b}$. Indeed, it is easy to see that there are distinct sequences which are 0 distance apart. A pseudo metric can always be turned into a metric on the *quotient space* obtained by identifying all points whose distance is zero. In this case, the quotient space is exactly the space obtained by identifying sequences which map to the same point under the semiconjugacy $h$ defined above, i.e. which give distinct expansions in base $\lambda$ and the metric then corresponds exactly to the Euclidean metric on the interval.

## 12.2 Symbolic coding

The arguments described above for the case study $f(x) = 10x \bmod 1$ can easily be generalized to any full branch piecewise expanding map and yield the following general result.

**Theorem 12.3.** *Let $f : I \to I$ be a full branch piecewise expanding map with $\ell$ branches. Then there exists a continuous surjective map*

$$h : \Sigma_\ell^+ \to I \quad \text{such that} \quad h \circ \sigma = f \circ h.$$

*In particular $f$ is transitive and $Per(f)$ is dense in $I$. The semi-conjugacy $h$ fails to be injective only on a countable set $\hat{I}^* \subset I$, formed by the boundary points of the partition elements and their pre-images, on which it is 2-1.*

The statements on transitivity and density of periodic orbits follow immediately from the fact that $h$ is continuous and surjective in which case it maps dense sets to dense sets. We sketch here the formalization of the argument described above First of all notice that that the assumptions that $\overline{f(I_j)} = I$ and that $|f'(x)| \geq \lambda > 1$ for all $x \in I_j$ imply that $f_j := f|_{I_j}$ is a homeomorphism onto its image and admits a continuous extension $\overline{f}_j$ to the closure $\overline{I}_j$ of $I_j$ so that $\overline{f}_j(\overline{I}_j) = I$. We can thus define a *multivalued* map $\overline{f} : I \to I$ by

$$\overline{f}(x) = \begin{cases} f_j(x) & \text{if } x \in I_j \text{ for some } j \in \{0, ..., \ell - 1\} \\ \overline{f}_i(x) \cup \overline{f}_j(x) & \text{if } x \in \overline{I}_i \cap \overline{I}_j \text{ for some } i, j \in \{0, ..., \ell - 1\} \end{cases}$$

Notice that if $f$ is continuous, then $\overline{f}$ is single valued everywhere and in fact $\overline{f} = f$. For any $\underline{a} \in \Sigma_\ell^+$ we then define

$$I_{\underline{a}} := \{x \in I : \overline{f^i}(x) \cap \overline{I}_{a_i} \neq \emptyset, \text{ for all } i = 0, 1, 2, ...\}$$

Notice that in the special case in which the closures of the intervals $I_j$ are disjoint, as in the previous chapter, we have $\overline{f} = f$ and the definition of $I_{\underline{a}}$ coincides exactly with the previous definition.

**Lemma 12.4.** *For each $\underline{a} \in \Sigma_\ell^+$, $I_{\underline{a}}$ is non-empty and consists of a single point.*

*Proof.* The argument is almost identical to that used in the proof of Lemma 11.3. The slight variation is that here we need to define

$$I_{a_0 \ldots a_n} := \{x \in I : \overline{f^i}(x) \cap \overline{I}_{a_i} \neq \emptyset, \text{ for all } i = 0, 1, 2, ..., n\}.$$

Then it is easy to see, following the same inductive argument as in the proof of Lemma , that $I_{a_0 \ldots a_n}$ is a closed interval and we conclude that $I_{\underline{a}}$ is non-empty and connected. Moreover, from the condition that $|f'(x)| \geq \lambda > 1$ for all $x \in I_i$ for all $i = 0, ..., \ell - 1$ we get

$$|I_{a_0 \ldots a_n}| \leq \lambda^{-(n+1)} \to 0 \tag{12.1}$$

as $n \to \infty$. Thus $I_{\underline{a}}$ is a single point. □

Lemma 12.4 allows us to define the map $h : \Sigma_\ell^+ \to I$ by $h(\underline{a}) = I_{\underline{a}}$. It follows immediately from the definition that $h$ is a semi-conjugacy and it is clearly surjective since every point belongs to at least on $I_{\underline{a}}$.

**Lemma 12.5.** *$h$ is continuous.*

*Proof.* By (12.1), for any $\epsilon > 0$ there exists $n_\epsilon$ sufficiently large so that for any $\underline{a} \in \Sigma_\ell^+$ we have $|I_{a_0 \ldots a_{n_\epsilon}}| \leq \lambda^{-n_\epsilon + 1} < \epsilon$. By Lemma 14.3, for this $n_\epsilon$ there exists a $\delta_{n_\epsilon} > 0$ so that $d(\underline{a}, \underline{b}) < \delta_{n_\epsilon}$ implies $a_i = b_i$ for all $i = 0, ..., n_\epsilon$ which implies that $h(\underline{a}), h(\underline{b})$ belong to the same interval $I_{a_0 \ldots a_{n_\epsilon}}$ which implies $|h(\underline{a}) - h(\underline{b})| < \epsilon$. □

**Lemma 12.6.** *$h$ is not injective on a countable set $\hat{I}^*$, on which it is 2-1.*

*Proof.* Let $I^* := \{x \in I : x \in \overline{I}_{j-1} \cap \overline{I}_j : i, j \in \{1, ..., \ell - 1\}\}$. denote the set of intersection points of the closures of the intervals $I_0, ..., I_{\ell-1}$ and $\hat{I}^* := \{x \in I : f^n(x) \in I^* \text{ for some } n \geq 0\}$ denote the set of all preimages of $I^*$. Clearly $I^*$ is a finite set and therefore $\hat{I}^*$ is a countable set. For any $x \in I \setminus \hat{I}^*$, by definition there exists a *unique* sequence $\underline{a} = a_0 a_1 a_2 ...$ such that $f^n(x) \in \overline{I}_{a_n}$ for all $n \geq 0$ and therefore $h(\underline{a}) = x$ and $h(\underline{b}) \neq x$ for all $\underline{b} \neq \underline{a}$. This shows that $h$ is injective on $I \setminus \hat{I}^*$. We remark in particular that $h$ is injective on the two endpoints of the interval $I$ on which the map $f$ is defined, without loss of generality we can assume these endpoints are 0 and 1. Let us denote by $\underline{a} = a_0 a_1 a_2 ....$ and $\underline{b} = b_0 b_1 b_2 ...$ the symbolic sequence of these endpoints. , i.e. $h(\underline{a}) = 0$ and $h(\underline{b}) = 1$. Notice that if the endpoints 0, 1 are fixed then these are simply the sequences $\underline{0} = 000...$ and $\underline{\ell - 1} = (\ell - 1)(\ell - 1)(\ell - 1)....$ but, depending on the signs of the derivatives on the two extreme branches we can also have $\underline{a} = 0(\ell - 1)(\ell - 1)(\ell - 1)...$ and $\underline{b} = (\ell - 1)(\ell - 1)(\ell - 1)...$ if 1 is a fixed point but 0 is not, $\underline{a} = 0000...$ and $\underline{b} = (\ell - 1)000...$ if 0 is a fixed point and 1 is not, or $\underline{a} = 0(\ell - 1)0(\ell - 1)...$ and $\underline{b} = (\ell - 1)0(\ell - 1)0...$ if neither 0 nor 1 are fixed points.

It just remains to show that $h$ is 2-1 on $\hat{I}^*$. Consider first $x \in I^*$. Then, by definition, $x \in \overline{I}_i \cap \overline{I}_j$ for some $i, j \in \{0, ..., \ell - 1\}$. Then there are clearly exactly two possible codings for this point: the first coding is given by the sequence starting with the digit $i$ followed the unique sequence associated to the endpoint of $I$ to which $\overline{f}_i$ maps the point, and the other given by the digit $j$ followed by the unique sequence corresponding to the endpoint of $I$ to which $\overline{f}_j$ maps the point. This shows that $h$ is 2-1 on $I^*$. Points in $\hat{I}^*$ have a unique combinatorics corresponding to the initial piece of orbit before thay land on $I^*$ at which point they have two possibe combinatorics. This completes the proof. □

## 12.3 Topological conjugacy

One of the key properties of topological conjugacy is the fact that it is an equivalence relation, which allows us for example to prove that two systems are conjugate by conjugating both to a third system, for example a symbolic model, as in the case of dynamically defined Cantor set. it is not immediately clear how to do this with semiconjugacies in general, though in the specific examples of $f(x) = 10x \mod 1$ studied above, we can get some insight. Indeed, it is easy to see that another map $g$ which is a nonlinear perturbation of $f$, in particular which still is a full branch map on 10 intervals, just like $f$, will have the same coding issues, in the sense that the lack of injectivity of the coding will occur exactly at the same points. This we will have two maps:

$$h_1 : \Sigma_{10}^+ \to [0,1] \quad \text{and} \quad h_2 : \Sigma_{10}^+ \to [0,1]$$

both of which are semiconjugacies. In principle they are not invertible, but amazingly in this case, the composition

$$h_1 \circ h_2^{-1}$$

is actually defined and is a bijection and in fact is a homeomorphism. Thus $f$ and $g$ are topologically conjugate.

Notice however that not all piecewise expanding map can be topologically conjugate, even if they have the same number of branches.

**Lemma 12.7.** *The two maps $f, g : [0,1] \to [0,1]$ given by*

$$f(x) = \begin{cases} 2x & \text{if } x < 0.5 \\ -2x+2 & \text{if } x \geq 0.5 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 2x & \text{if } x < 0.5 \\ 2x-1 & \text{if } x \geq 0.5 \end{cases} \tag{12.2}$$

*are not topologically conjugate.*

*Proof.* A topological conjugacy consists of a homeomorphism $h : [0,1] \to [0,1]$ which in particular must send endpoints to endpoints. Moreover it must send orbits to orbits and thus in particular fixed points to fixed points. However the two fixed points of $g$ are the two endpoints whereas the two fixed points of $f$ are one of the endpoints and the other is in the interior, thus $h$ cannot be a conjugacy. □

Let $f : I \to I$ be a full branch piecewise expanding map with $\ell$ branches. Then by definition $f|_{I_i}$ is a diffeomorphism onto its image and thus can be either orientation preserving and orientation reversing. It turns out that the information about the topological conjugacy class is completely contained in the number of branches and their orientation.

**Theorem 12.8.** *Let $f, g$ be two full branch piecewise expanding maps with the same number of branches and such that corresponding branches have the same orientation. Then $f, g$ are topologically conjugate.*

*Proof.* From the previous construction we have that every point of $I \setminus \hat{I}^*$ has a unique combinatorics and that every point in $\hat{I}^*$ has exactly two possible combinatorial associated sequences. Since $f, g$ have the same number of branches and corresponding branches have the same orientation, the endpoints have the same unique combinatorics. Thus we can clearly map endpoints to endpoints (whether they are fixed or not). Moreover, since the branches all have the same orientation, their endpoints map to the corresponding endpoints of $I$ and thus all points of $I^*$ have the same combinatorics for $f$ and for $g$, and the same is also true for all pre images of $I^*$. Thus we can simply map all points to the corresponding point with the same combinatorics. This gives an order preserving bijection which conjugates the dynamics of $f, g$.

It just remains to prove that $h, h^{-1}$ are continuous. This follows by essentially the same arguments used above, we sketch the proof and leave the details as an exercise. Let $x \in I$ and $\epsilon > 0$. Then to prove that $|h(x) - h(y)| < \epsilon$ it is sufficient to have that $h(x), h(y) \in I'_{a_0 a_1 \ldots a_{n_\epsilon}}$ for some sufficiently large $n_\epsilon$. Since $h$ preserves the combinatorics of points, it is sufficient to have that $x, y \in I_{a_0 a_1 \ldots a_{n_\epsilon}}$ and for that it is sufficient that $|x - y| < \delta$ for some sufficiently small $\delta$. Notice that if $x \in \hat{I}^*$ then it has two possible associated combinatorial sequences and the correct one to choose in the argument simply depends on whether $y$ is on the left or on the right of $x$. □

**Corollary 12.9.** *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be $C^1$ and suppose there exists $\lambda > 1$ such that $f'(x) \geq \lambda$ for all $x \in S^1$. Then $f$ is structurally stable.*

*Proof.* Notice that any $C^1$ map of $\mathbb{S}^1$ is a *covering map* and can therefore be represented as a full branch map on the unit interval $[0, 1]$ with the endpoints identified. The number of branches corresponds exactly to the *degree* of the map. Moreover any small $C^1$ perturbation $g$ of $f$ will also be a full branch map on $\mathbb{S}^1$ with the same number of branches and will also satisfy the expanding condition. Thus $f, g$ can both be represented as piecewise expanding full branch maps with the same number of branches with the same orientation and thus are topologically conjugate. Therefore $f$ is structurally stable. □

# Chapter 13

# The quadratic family

We conclude this chapter by introducing an important family of maps $f_a : \mathbb{R} \to \mathbb{R}$, the so-called *quadratic family*, defined by

$$f_a(x) = x^2 + a.$$

Notice that for large negative of the parameter $a$ there exists an interval $I$ and two closed disjoint subintervals $I_0, I_1 \subset I$ on which $f$ is expanding and such that $f(I_0) = f(I_1) = 1$. Thus for these parameters the maps have invariant Cantor sets as described in the previous chapter. For parameters $a < -2$ but close to $-2$ we still have two closed disjoint subintervals but the map is no longer expanding. nevertheless it is possible to show that in this case the weaker expansivity condition (11.3) holds, and thus we continue to have invariant Cantor sets. In this section we will focus on the parameter $a = -2$ in which we have a full branch map with two intervals whose closures are not disjoint and which also clearly cannot satisfy the expansivity condition (11.3) since it has a point where $f'(x) = 0$. Notice in particular that the symbolic coding argument cannot be applied, at least not directly, in this case, since we used in an esential way the expansivity properties of the map to to show that the symbolic coding was injective.

**Proposition 13.1.** *The maps $f : [-2, 2] \to [-2, 2]$ and $g : [0, 1] \to [0, 1]$ defined by*

$$f(x) = x^2 - 2 \quad \text{and} \quad g(z) = \begin{cases} 2z, & 0 \leq z < \frac{1}{2} \\ 2 - 2z, & \frac{1}{2} \leq z \leq 1. \end{cases}$$

*are topologically conjugate.*

The map $f(x) = x^2 - 2$ is sometimes called the *Ulam-von Neumann* map.

*Proof.* This is one of the very exceptional situations in which we can find a conjugacy completely explicitly. Define the map $h : [0,1] \to [-2,2]$ by

$$h(z) = 2\cos \pi z.$$

$h$ is clearly a (orientation reversing) homeomorphism and so we just need to show that it is a conjugacy, i.e. that it satisfies the conjugacy equation $f \circ h = h \circ g$. On one hand we have

$$f(h(z)) = f(2\cos \pi z) = (2\cos \pi z)^2 - 2 = 4\cos^2 \pi z - 2 = 2(2\cos^2 \pi z - 1) = 2\cos 2\pi z.$$

On the other hand we have, for $z \in [0, 1/2)$,

$$h(g(z)) = h(2z) = 2\cos \pi 2z$$

and, for $z \in [1/2, 1]$,

$$h(g(z)) = h(2-2z) = 2\cos \pi(2-2z) = 2\cos(2\pi - 2\pi z) = 2\cos(-2\pi z) = 2\cos 2\pi z$$

This proves the conjugacy. □

As an immediate corollary of the topological conjugacy we get

**Corollary 13.2.** *The map $f : [-2,2] \to [-2,2]$ is transitive and has a dense set of periodic orbits.*

Notice that $h : [0,1] \to [-2,2]$ is a homeomomorphism but if we restrict to the open interval $(0,1)$ we actually get a $C^1$ diffeomorphism $h : (0,1) \to (-2,2)$. It does not extend as a $C^1$ diffeomorphism to the closed interval $[0,1]$ beacuse $h'(0) = h'(1) = 0$. However this means that $f, g$ are *almost* $C^1$ conjugate, and indeed, to all effects and purposes they are. This means that the conjugacy preserves significantly more structure than just a topological conjugacy. As a first interesting corollary we mention the following.

**Proposition 13.3.** *For every periodic point $p \in (-2,2)$ of period $n$, $|(f^n)'(p)| = 2^n$.*

*Remark* 13.4. Notice that by the topological conjugacy with the tent map, $f$ has a dense set of periodic points which means that there are periodic points arbitrarily close to the critical point. For a periodic point $p$ of period $n$ we have

$$|(f^n)'(p)| = |f'(p)f'(f(p)) \cdots f'(f^{n-1}(p))|$$

thus the result says that for any periodic orbit the derivatives compensate each other exactly. In particular, any orbit for which some point lies very close to the critical point must have a very high period in order to compensate the small derivative near the critical point.

*Proof.* The assumption that $p \in (-2, 2)$ implies that the entire orbit lies in $(-2, 0) \cup (0, 2)$. Indeed, if some iterate of $p$ falls on the critical point at 0 or on one of the endpoints $\pm 2$, it would then fall onto the fixed point at 2 contradicting the assumption that $p$ is a periodic orbit and that $p \in (-2, 2)$. Since the entire orbit lies in $(-2, 2)$ we can use the fact that the conjugacy $h$ is $C^1$ and that $p$ is a fixed point for $f^n$ and is therefore mapped to a fixed point $q$ for $g^n$, to get that the derivative of $f^n$ at $p$ is the same as the derivative of $g^n$ at $q$ which is necessarily $2^n$. $\square$

# Part V

# Minimal Dynamics

# Chapter 14

# Minimal Homeomorphisms

The dynamical systems we have studied so far exhibit a variety of dynamical behaviour but are all essentiallyu *simple* in the sense that every initial condition eventually either diverges to infiinity or converges to some fixed point. We now intrdoduce a couple of examples of systems which have no fixed or periodic points but nevertheless exhibit nontrivial *recurrence*.

**Definition 14.1.** Let $X$ be a metric space and $f : X \to X$ be a map. We say that a point $x \in X$ has a *dense* orbit if $\omega(x) = X$. We say that $f$ is *minimal* if every $x \in X$ has a dense orbit in $X$.

Trivial examples of minimal systems can be obtained when for example $X$ is just a single point or a finite set of points which are cyclically permuted. However there are several other very interesting and highly non-trivial examples of such dynamical behaviour. We will present here two important examples.

## 14.1 Circle rotations

### 14.1.1 An application to Number Theory

Many results in dynamics have application to other areas of mathematics. In particular there is a very strong connection between certain dynamical systems and certain kinds of results in number theory. We present here a first example of such a situation.

**Proposition 14.2.** *Let $k \in \mathbb{N}$ be a natural number other than a power of 10. Then, for any given finite sequence of digits $a_0 a_1 \ldots a_\ell$, there exists an $n \in \mathbb{N}$ such that the initial digits of the number $k^n$ coincide exactly with the given sequence.*

*Proof.* The statement is equivalent to saying that there is some $m, n \in \mathbb{N}$ and some sequence $a_{\ell+1} \dots a_{\ell+m}$ such that

$$k^n = a_0 \dots a_\ell a_{\ell+1} \dots a_{\ell+m} = a_0 \dots a_\ell \times 10^m + a_{\ell+1} \dots a_{\ell+m}.$$

Letting $p = a_0 \dots a_\ell$ this is equivalent to saying that there exsts some $n, m, q \in \mathbb{N}$ with $q < 10^m$ such that

$$k^n = 10^m p + q$$

which is equivalent to saying that there exists some $m, n \in \mathbb{N}$ such that

$$10^m p < k^n < 10^m (p+1). \tag{14.1}$$

Therefore we just need to show that there exist $m, n$ satisfying (14.1). Taking logs (base 10) this is equivalent to showing

$$m + \log_{10} p < n \log_{10} k < m + \log_{10}(p+1)$$

which is equivalent to showing that there exist $n, m \in \mathbb{N}$ such that

$$\log_{10} p < n \log_{10} k - m < \log_{10}(p+1).$$

This is saying that there exists some $n$ such that the *distance of* $n \log_{10} k$ *to the nearest integer* is contained in the interval $(\log_{10} p, \log_{10}(p+1))$. Notice that this is a interval of size $\log_{10}[(p+1)/p] \leq \log_{10} 2 < 1$ since $p \geq 1$ and $\log_{10}[(p+1)/p]$ is monotonically decreasing with $p$. Therefore it is sufficient to show that that *fractional part*

$$n \log_{10} k \mod 1$$

of $n \log_{10} k$ contained in the interval $(\log_{10} p, \log_{10}(p+1)) \subset (0,1)$ for some $n \geq 1$. Indeed, notice that letting

$$x_n = n \log_{10} k \mod 1 = \log_{10} k + \cdots + \log_{10} k \mod 1$$

the sequence $\{x_n\}_{n=0}^{\infty}$ is nothing less than the orbit of a point ($x_0 = 0$) under a "rotation" by the angle $\log_{10} k$ if we identify the interval $[0,1)$ with the circle $\mathbb{S}^1$ in the obvious way.

It is therefore sufficient to show that $\log_{10} k$ is irrational to conclude that its orbit is dense and in particular the points of the "orbit" $x_n$ must fall (infinitely many times) in the intervals $(\log_{10} p, \log_{10}(p+1))$. To show that $\log_{10} k$ is irrational, suppose by contradiction that there exist integers $p, q$ such that $\log_{10} k = p/q$. Then this is equivalent to $q \log_{10} k = p$ or $k^q = 10^p$. Clearly the last equality can only hold if $k$ is a power of 10. □

## 14.2 The adding machine

Irrational circle rotations are a special case of *translations on compact groups*, which often have similar dynamical properties. We give here another example of such a translation in a more abstract setting. We first need to define a very interesting metric space.

### 14.2.1 Symbolic spaces

Let $\ell \geq 2$ and

$$\mathcal{A} := \{0, ...., \ell - 1\}$$

be a finite *alphabet* of $\ell$ symbols, and let

$$\Sigma_\ell^+ := \{\underline{a} = a_1 a_2 a_3 ... : a_i \in \mathcal{A}\}$$

be the set of all (one-sided) infinite *words* made up of this alphabet.

**Exercise 14.2.1.** Show that $\Sigma_\ell^+$ is uncountable.

For two sequence $\underline{a}, \underline{b} \in \Sigma_\ell^+$ we let

$$d(\underline{a}, \underline{b}) = \sum_{i=1}^{\infty} \frac{|a_i - b_i|}{\ell^i}$$

**Exercise 14.2.2.** Show that $d$ is a metric[1] on $\Sigma_\ell^+$.

**Exercise 14.2.3.** Show that $(\Sigma_\ell^+, d)$ is a Cantor set, i.e. it is (sequentially) compact, totally disconnected and perfect (no isolated points).

**Exercise 14.2.4.** Show that the function

$$d(\underline{a}, \underline{b}) = \left| \sum_{i=1}^{\infty} \frac{a_i - b_i}{\ell^i} \right|$$

is a *pseudo metric* but not a metric, i.e. it satisfies conditions ii) and iii) in the definition of a metric, but not i). It can however become a metric by making certain identifications on the space. Can you describe these identifications? Suppose $\ell = 10$. Can you define a correspondence between $\Sigma_{10}^+$ and all real numbers in the interval $[0, 1]$ using their decimal expansions? What is the relation between the Euclidean metric on $[0, 1]$ and the metric on $\Sigma_{10}^+$ defined above?

[1]Recall that $d(x, y)$ is a *metric* on $X$ if it satisfies the following conditions: i) $d(x, y) \geq 0 \ \forall \ x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$; ii) $d(x, y) = d(y, x)$ for all $x, y \in X$; iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

A key property of this metric is that two sequences are close if and only their terms coincide for a sufficiently large initial block. We will use this property repeatedly below.

**Lemma 14.3.** *For every $\epsilon > 0$ there exists $n_\epsilon > 0$, with $n_\epsilon \to \infty$ when $\epsilon \to 0$, such that if $\underline{a}, \underline{b} \in \Sigma_2^+$ satisfy $a_i = b_i$ for all $i = 0, ..., n_\epsilon$ then $d(\underline{a}, \underline{b}) < \epsilon$. Conversely, for every $n > 0$ there exists $\epsilon_n > 0$, with $\epsilon_n \to 0$ when $n \to \infty$, such that if $d(\underline{a}, \underline{b}) < \epsilon_n$ then $a_i = b_i$ for all $i = 0, ..., n$.*

*Proof.* Given $\epsilon > 0$ let $n_\epsilon$ be sufficiently large so that $\ell^{-n_\epsilon} < \epsilon$. Then, if $a_i = b_i$ for all $i = 1, ..., n_\epsilon$ we have

$$d(\underline{a}, \underline{b}) := \sum_{i=1}^{\infty} \frac{|a_i - b_i|}{\ell^i} = \sum_{i=n_\epsilon+1}^{\infty} \frac{|a_i - b_i|}{\ell^i} \le \sum_{i=n_\epsilon+1}^{\infty} \frac{1}{\ell^i} = \frac{1}{\ell^{n_\epsilon}} < \epsilon.$$

Conversely, given $n > 0$, let $\epsilon_n = \ell^{-n}$ and assume that $d(\underline{a}, \underline{b}) < \epsilon_n$. Suppose by contradiction that there exists some $j \in \{1, ...., n\}$ such that $a_j \neq b_j$. Then $|a_j - b_j| = 1$ and therefore

$$d(\underline{a}, \underline{b}) := \sum_{i=1}^{\infty} \frac{|a_i - b_i|}{\ell^i} \ge \frac{1}{\ell^j} \ge \frac{1}{\ell^n} = \epsilon_n.$$

This contradicts the assumption that $d(\underline{a}, \underline{b}) < \epsilon_n$ and therefore shows that $a_i = b_i$ for all $i \le n$. □

### 14.2.2 The adding machine

We define the **adding machine** or **odometer** $\tau : \Sigma_\ell^+ \to \Sigma_\ell^+$,

$$\tau(x_1 x_2 x_3 ...) = y_1 y_2 y_3 ...$$

by the operation of *add 1 and carry* as follows. Let $\underline{x} \in \Sigma_\ell^+$. First of all, if $x_i = \ell - 1$ for all $i \ge 1$, i.e. if $\underline{x}$ is the constant sequence $(\ell-1)(\ell-1)(\ell-1)...$, then we let $y_i = 0$ for all $i \ge 1$. Otherwise we let

$$i_0(\underline{x}) := \min\{i \ge 1 : x_i \neq \ell - 1\}$$

If $i_0(\underline{x}) = 1$ then $x_1 \in \{0, ..., \ell-2\}$ and we let $y_1 = x_1 + 1$ and $y_i = x_i$ for all $i > 1$. If $i_0(\underline{x}) > 0$ then we have $x_i = \ell - 1$ for all $i_0 > i \ge 1$ and thus we let $y_i = 0$ for all $i_0 > i \ge 1$, let $y_{i_0} = x_{i_0} + 1$, and $y_i = x_i$ for all $i > i_0$.

**Lemma 14.4.** *The map $\tau : \Sigma_\ell^+ \to \Sigma_\ell^+$ is a homeomorphism.*

*Proof.* To see that $h$ is injective let $\underline{x} \neq \underline{x}'$. Then we must have $x_i \neq x'_i$ for some $i \geq 0$. Let $i_1 := \min\{i \geq 0 : x_i \neq x'_i\}$. By the definition of $\tau$ we then necessarily have $y_i \neq y'_i$. Therefore $\tau$ is injective. To see that it is surjective we just define the inverse map $\tau^{-1}(y_0 y_1 y_2 ...) = x_0 x_1 x_2 ....$ explicitly as follows. If $y_i = 0$ for all $i \geq 0$ then let $x_0 = \ell - 1$ for all $i \geq 0$. Otherwise, let $j_0 := \min\{j \geq 0 : y_j \neq 0\}$. If $j_0 = 0$ then we just let $x_0 = y_0 - 1$ and $x_i = y_i$ for all $i > 0$. Otherwise, let $x_i = \ell - 1$ for all $j_0 > i \geq 1$, $x_{i_0} = y_{i_0} - 1$, and $x_i = y_i$ for all $i > i_0$. This shows that $\tau$ is surjective.

To show that $\tau$ is continuous, recall from the definition of the metric $d$ that for every $\delta > 0$ there exists $i_\delta$ such that $|\underline{x} - \underline{x}'| < \delta$ if $x_i = x'_i$ for all $i \leq i_\delta$. By the definition of $\tau$, letting $\tau(\underline{x}) = \underline{y}, \tau(\underline{x}') = \underline{y}'$, this implies $y_i = y'_i$ for all $i \leq i_\delta$. Therefore $\tau$ is uniformly continuous. The (uniform) continuity of $\tau^{-1}$ is proved in the same way. □

**Theorem 14.5.** *The dynamics of* $\tau$ *is "minimal", i.e.* $\omega(\underline{x}) = \Sigma_\ell^+$ *for all* $\underline{x} \in \Sigma_\ell^+$.

*Proof.* Let $\underline{x} \in \Sigma_\ell^+$. We need to show that the forward orbit $\mathcal{O}^+(\underline{x})$ accumulates every point in $\Sigma_\ell^+$, i.e. for every $\underline{z} \in \Sigma_\ell^+$ and every $\varepsilon > 0$ there exists $n \geq 0$ such that $d(\tau^n(\underline{x}), \underline{z}) < \varepsilon$. By the definition of the metric, there exists an integer $n_\epsilon$, depending only on $\epsilon$, such that $d(\tau^n(\underline{x}), \underline{z}) < \varepsilon$ if $\tau^n(\underline{x})$ and $\underline{z}$ coincide for at least the initial $n_\epsilon$ terms of the sequences.

We prove the statement first for the particular initial condition $\underline{x} = \underline{0} = 0000....$ Then the first few iterates are as follows $000\ldots \to 100\ldots \to 200\ldots \to \ldots \to (n-1)00\ldots 010\ldots \to 110\ldots 210\ldots$. Considering the first $n_\epsilon$ digits in the sequence of iterates, it follows that all possible combination of digit will have occurred after at most $\ell^{n_\epsilon}$ iterations. Since we can take $\epsilon$ arbitrary, this implies that the orbit of $\underline{0}$ is dense. For an arbitrary initial condition $\underline{x} = x_1 x_2 \ldots$ notice that after at most $\ell$ iterations, the digit 0 will appear in the first position, after at most $\ell^2$ iterations the digit 0 wil apear in *both* the first and second place, and continuing in this way, after at most $\ell^{n_\epsilon}$ iterations, the digit 0 will appear in all of the first $n_\epsilon$ places. Therefore we have some iterate of $\underline{x}' = \tau^n(\underline{x})$ with $n \leq \ell^{n_\epsilon}$ such that $x'_i = 0$ for all $i \leq n_\epsilon$. It follows then, by the argument used previously that all possible combination of the first $n_\epsilon$ digits will occur with a maximum of a further $\ell^{n_\epsilon}$ iterations. Since $\epsilon$ is arbitrary this implies again that the orbit of $\underline{x}$ is dense. □

# Chapter 15

# Translations and Circle Homeomorphisms

We now begin a study of an important class of discrete dynamical systems: *circle homeomorphisms*. Beacuse these are homeomorphisms of compact spaces they cannot be either fully contracting or fully expanding but rather have a kind of neutral behaviour on average. Special cases of circle homeomorphisms are rigid circle rotations which are also special cases of translations on compact groups. In this chapter we will study such kinds of systems. We will show that in many cases we have one of two kinds of behaviour for all orbits. We recall that the (forward) orbit of a point $x_0 \in X$ is *dense* in $X$ if $\omega(x_0) = X$.

**Theorem 15.1.** *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a homeomorphism.*
*If $f$ has a periodic orbit, then every point is asymptotic some some periodic orbit.*
*If $f$ has a dense orbit, then every orbit is dense.*

At the end of the chapter we will also how that there exist examples of circle homeomorphisms which have neither periodic orbits nor dense orbits. A classical result of Denjoy, which we will not prove here, says that such counterexamples cannot occur if $f$ is a $C^2$ diffeomorphism, in which case we have only the two possibilities stated in the Theorem.

## 15.1 Circle homeomorphisms with periodic points.

We consider first of all the case in which $f$ is orientation preserving, i.e. monotone increasing. Recall first of all that if $J$ is a closed interval and $f : J \to J$ is an orientation preserving continuous injective map, then every orbit is asymptotic to a fixed point. This is of course not necessarily true if we replace $J$ by $\mathbb{S}^1$ (as for example irrational circle rotations are monotone increasing but have no periodic points. However, let us *assume* that $f : \mathbb{S}^1 \to \mathbb{S}^1$ is orientation preserving and has at least one periodic point $p$ of period $k$. Let us assume first that $k = 1$ so that $p$ is a fixed point. Then we can identify the circle with the half-closed interval $(p, p]$ and $f$ is monotone increasing on this interval. By the same arguments it follows that every orbit must be asymptotic to some fixed point for $f$. If $p$ is a periodic point of period $k$ then similarly $p$ is a fixed point for $f^k$ which is monotone increasing on the interval $(p, p]$ and we get that every orbit is asymptotic to a periodic point of period $k$.

Suppose now that $f$ is orientation reversing. Notice first of all that this implies that it always has exactly two fixed points as can be easily seen just by considering the graph of $f$. Therefore $f^2$ is orientation preserving and also has (at least two fixed points, and thus we proceed as above.

This completes the proof that if $f$ contains at least one periodic orbit, than every point is asymptotic to a periodic orbit.

## 15.2 Rotation number

The dynamics of homeomorphisms without periodic points is significantly more complicated and requires introducing some additional notions and tools. The most important is the notion of *rotation number* of $f$ which measures the "average" amount of rotation of points under iteration by $f$. To define the rotation number we first need to introduce the concept of a *lift* of a circle homeomorphism.

Recall that $\mathbb{S}^1$ can be represented as the quotient $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ which is by definition the set of equivalence classes of real numbers whose difference is an integer. We let

$$\pi : \mathbb{R} \to \mathbb{S}^1$$

denote the natural *projection map* onto this quotient which maps each real number to the point on $\mathbb{S}^1$ corresponding to its fractional part:

$$\pi(x) = x \mod 1 = x - \lfloor x \rfloor$$

where

$$\lfloor x \rfloor := \max\{k \in \mathbb{Z} : k \leq x\}.$$

This projection is clearly continuous and surjective.

**Definition 15.2.** Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a circle homeomorphism. We say that $F : \mathbb{R} \to \mathbb{R}$ is a *lift* of $f$ if

$$\pi \circ F = f \circ \pi.$$

*Remark* 15.3. Notice that this condition looks exactly like the conjugacy condition, except that $\pi$ is only *surjective*, and not bijective. We say that $\pi$ is a *semiconjugacy* between $F$ and $f$.

*Example* 23. Let $f = R_\alpha : \mathbb{S}^1 \to \mathbb{S}^1$ be a circle rotation

$$f(x) = x + \alpha \mod 1$$

Let $k \in \mathbb{Z}$ and define the function $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = x + \alpha + k.$$

Then

$$\pi(F(x)) = \pi(x+\alpha+k) = x+\alpha+k \mod 1 = x+\alpha \mod 1 = \pi(x)+\alpha \mod 1 = f(x).$$

So $F$ is a lift of $f$.

**Proposition 15.4.** *For any homeomorphism $f : \mathbb{S}^1 \to \mathbb{S}^1$*

1. *There exists a lift $F : \mathbb{R} \to \mathbb{R}$;*
2. *any lift $F$ of $f$ is a homeomorphism;*
3. *if $F, G$ are two lifts of $f$, there exists $k \in \mathbb{Z}$ such that $F - G = k$;*
4. *if $f$ is orientation preserving and $F$ is a lift of $f$, then $F(x+n) = F(x) + n$ for any $x \in \mathbb{R}$ and any $n \in \mathbb{Z}$.*

*Proof.* Let

$$F(x) := f(x - \lfloor x \rfloor) + \lfloor x \rfloor)$$

Then it is easy to verify that $F$ is a lift of $f$. We leave the other conditions as exercises. □

**Proposition 15.5.** *Let $f : \mathcal{S}^1 \to \mathcal{S}^1$ be an orientation-preserving homeomorphism and $F : \mathbb{R} \to \mathbb{R}$ a lift of $f$. Then, for every $x \in \mathbb{R}$, the limit*

$$\lim_{n\to\infty} \frac{F^n(x) - x}{n}$$

*exists and is independent of the point $x$. We denote this limit by $\rho(F)$. Moreover, the number*

$$\pi(\rho(F))$$

*is independent of the lift $F$.*

**Definition 15.6.** Since $\pi(\rho(F))$ does not depend on the specific choice of lift $F$ we can define

$$\rho(f) := \pi(\rho(F))$$

for any lift $F$. $\rho(f)$ is called the *rotation number* of $f$.

*Example* 24. The rotation number measures the *average rotation* of points under iteration of $f$. If $f(x) = x + \alpha$ mod 1 is a rigid circle rotation, then $\rho(f) = \alpha$.

Before starting the proof of the Proposition we state and prove a simple preliminary Lemma.

**Lemma 15.7.** *Let* $\{c_n\}$ *be a* subadditive *sequence of positive numbers (i.e.* $c_{m+n} \leq c_m + c_n$ *for all* $m, n \in \mathbb{N}$*). Then*

$$\lim_{n\to\infty} \frac{c_n}{n} = \inf \left\{ \frac{c_n}{n} : n \in \mathbb{N} \right\}.$$

*In particular the limit exists.*

*Proof.* For any positive integers $k, n$ we can write $n = qk + r$ for some $q \in \mathbb{N} \cup \{0\}$ and $r \in \{0, ..., q-1\}$. Then we have

$$\frac{c_n}{n} \leq \frac{c_{qk} + c_r}{qk + r} \leq \frac{qc_k + c_r}{qk + r}.$$

For arbitrary fixed $k$ we therefore have

$$\limsup_{n\to\infty} \frac{c_n}{n} \leq \frac{c_k}{k}.$$

and so, as $n \to \infty$ we have that $q \to \infty$ and thus, since $k$ is arbitrary, we have

$$\limsup_{n\to\infty} \frac{c_n}{n} \leq \inf \left\{ \frac{c_n}{n} : n \in \mathbb{N} \right\} \leq \liminf_{n\to\infty} \frac{c_n}{n}.$$

□

*Proof of Proposition 15.5.* We fix some arbitrary $x \in \mathbb{S}^1$. Letting $a_n := F^n(x) - x$ we will show that the sequence $c_n = a_n + 1$ is subadditive and thus, by Lemma 15.7 we have

$$\lim_{n\to\infty} \frac{F^n(x) - x}{n} = \lim_{n\to\infty} \frac{a_n}{n} = \lim_{n\to\infty} \frac{a_n + 1}{n} = \lim_{n\to\infty} \frac{c_n}{n} = \inf \left\{ \frac{c_n}{n} : n \in \mathbb{N} \right\} \in [0, \infty).$$

To show that $c_n$ is subadditive, notice that

$$a_{m+n} = F^{m+n}(x) - x = F^m(F^n(x)) - F^n(x) + a_n. \tag{15.1}$$

To bound the right hand side notice first of all that

$$\lfloor a_n \rfloor \leq F^n(x) - x < \lfloor a_n \rfloor + 1$$

and in particular

$$F^n(x) < \lfloor a_n \rfloor + x + 1$$

therefore, using also the fact that $F$ is orientation preserving and thus monotone increasing, and the periodicity of $F$ we have

$$F^m(F^n(x)) < F^m(\lfloor a_n \rfloor + x + 1) = F^m(\lfloor a_n \rfloor + x) + 1. \tag{15.2}$$

To bound the right hand side notice that

$$F^m(x + \lfloor a_n \rfloor) - (x + \lfloor a_n \rfloor) = F^m(x) + \lfloor a_n \rfloor - (x + \lfloor a_n \rfloor) = F^m(x) - x = a_m$$

and therefore

$$F^m(x + \lfloor a_n \rfloor) \leq a_m + x + \lfloor a_n \rfloor \tag{15.3}$$

Substituting (15.3) into (15.2) we get

$$F^m(F^n(x)) < a_m + x + \lfloor a_n \rfloor + 1$$

Substituting this into (15.1) we get

$$a_{m+n} < a_m + x + \lfloor a_n \rfloor + 1 - F^n(x) + a_n = a_m + a_n + 1 + \lfloor a_n \rfloor - (F^n(x) - x) \leq a_m + a_n + 1$$

Therefore the sequence $c_n = a_n + 1$ satisfies

$$c_{n+m} = a_{n+m} + 1 < a_m + a_n + 1 < (a_m + 1) + (a_n + 1) = c_n + c_m$$

and is therefore subadditive. We therefore have that the limit exists. We now show that this limit doesn't depend on the point $x$. Let $x, y \in \mathbb{R}$. Then we can choose some integer $k$ such that $|x - y| \leq k$. Moreover, by the periodicity of $F$ we have

$$F(x) \leq F(y + k) = F(y) + k \quad \text{and} \quad F(x) \geq F(y - k) = F(y) - k$$

and therefore $|F(x) - F(y)| \leq k$. Thus, inductively, we get

$$|F^n(x) - F^n(y)| \leq k$$

for any $n \in \mathbb{N}$. It follows that

$$\left| \frac{(F^n(x) - x)}{n} - \frac{(F^n(y) - y)}{n} \right| = \left| \frac{(F^n(x) - F^n(y)) + (y - x)}{n} \right| \leq \frac{2k}{n}.$$

This converges to 0 as $n \to \infty$ and so the limits are the same. Finally, we just need to show that for two lifts the limit can only differ by an integer. Since any two lifts differ only by an integer translation, any other lift has to be of the form $G = F + k$ for some integer $k$. Therefore $\rho(G) = \rho(F) + k$ and $\pi(\rho(G)) = \pi(\rho(F))$. □

### 15.2.1 Homeomorphisms with rational rotation number

**Proposition 15.8.** *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a homeomorphism. Then $\rho(f)$ is rational if and only if $f$ has at least one periodic orbit.*

*Proof.* Suppose first that $\rho(f) = 0$. We show that $f$ has a fixed point. Suppose by contradiction that $f$ does not have a fixed point. Then for any lift $F$ of $f$ and any $x \in \mathbb{R}$ we have

$$F(x) - x \in \mathbb{R} \setminus \mathbb{Z}$$

since otherwise $f$ would have a fixed point (check!). Since $F$ is continuous, this means that $F(x) - x$ is contained in some component of $R \setminus \mathbb{Z}$ for all $x \in \mathbb{R}$, i.e. there exists an integer $k$ such that

$$k < F(x) - x < k + 1$$

for all $x \in \mathbb{R}$. On the other hand $F$ is periodic and therefore the range of $F$ is completely determined by the range of $F$ on the unit interval $[0, 1]$. Since $[0, 1]$ is compact and $F$ is continuous, the range of $F$ is compact and therefore there exists $\epsilon > 0$ such that

$$k + \epsilon \le F(x) - x \le k + 1 - \epsilon.$$

Writing

$$F^n(x) - x = \sum_{i=0}^{n-1}[F^{i+1}(x) - F^i(x)] = \sum_{i=0}^{n-1}[F(F^i(x)) - F^i(x)]$$

it then follows that

$$k + \epsilon \le \frac{F^n(x) - x}{n} \le k + 1 - \epsilon.$$

and therefore

$$\rho(f) = \lim_{n\to\infty} \frac{F^n(x) - x}{n} \mod 1 \in [\epsilon, 1 - \epsilon]$$

which contradicts $\rho(f) = 0$. Thus $f$ must gave a fixed point. Now suppose that $\rho = p/q \in \mathbb{Q}$. If $F$ is a lift of $f$ the $F^q$ is a lift of $f^q$ and therefore we have

$$\rho(f^q) = \lim_{n\to\infty} \frac{(F^q)^n(x) - x}{n} \mod 1 = q \lim_{n\to\infty} \frac{F^{qn}(x) - x}{qn} = q\rho(f) \mod 1 = p \mod 1 = 0.$$

Thus, by the argument above, $f^q$ has a fixed point, and so $f$ has a periodic point of period $q$. This completes the proof in one direction, i.e. if $\rho(f)$ is rational then $f$ has at least a periodic point.

We leave the converse as an exercise. Notice that it is sufficient to show that the rotation number of this periodic point is rational, since we have proved above that the rotation number is independent of the point. □

## 15.3 Irrational rotation number

**Theorem 15.9** (Poincaré, 1900's)**.** *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a homeomorphism with irrational rotation number $\rho(f) = \alpha$. Then $f$ is topologically semi-conjugate to a rigid rotation $f_\alpha : \mathbb{S}^1 \to \mathbb{S}^1$: there exists a continuous surjective map $h : \mathbb{S}^1 \to \mathbb{S}^1$ such that $h \circ f = f_\alpha \circ h$. If moreover there exist a orbit for $f$ which is dense in $\mathbb{S}^1$ then $f$ is topologically conjugate to $f_\alpha$.*

The key technical step in this result is to show that the orbits of $f$ have the same order as the orbits of the rigid rotation $f_\alpha$. We formalize this notion as follows.

**Proposition 15.10.** *Let $F$ be a lift of a circle homeomorphism $f$ with irrational rotation number $\alpha$. Then, for every $x \in \mathbb{R}$ and any $n_1, n_2, m_1, m_2 \in \mathbb{Z}$ we have*

$$F^{n_1}(x) + m_1 < F^{n_2}(x) + m_2 \quad \textit{if and only if} \quad n_1\alpha + m_1 < n_2\alpha + m_2. \tag{15.4}$$

*Proof.* We assume that $n_1 \neq n_2$ otherwise there is nothing to prove. Notice first of all that the left hand side of (15.4) in principle depends on $x$ whereas the right hand side does not. We therefore begin by showing that in fact the left hand side does not depend on $x$ in the sense that it either holds for all $x \in \mathbb{R}$ or does not hold for any $x \in \mathbb{R}$. Since $F$ is continuous it is sufficient to show that we can never have an equality $F^{n_1}(x)+m_1 = F^{n_2}(x)+m_2$ for some $x \in \mathbb{R}$ so that the inequality is either always satisfied or never satisfied. Assuming by contradiction that we have equality, this implies $F^{n_1}(x) = F^{n_2}(x) + m_2 - m_1 = F^{n_2}(x + m_2 - m_1)$ and therefore $F^{n_1-n_2}(x) = x + m_2 - m_1$ but this would imply that $\pi(x)$ is a periodic point and this contradicts the assumption that the rotation number if irrational.

Now suppose that the left hand side of (15.4) holds for all $x \in \mathbb{R}$. We distinguish two cases. If $n_1 > n_2$ we have

$$F^{n_1-n_2}(x) < x + m_2 - m_1$$

for any $x \in \mathbb{R}$ and therefore

$$F^{2(n_1-n_2)}(x) = F^{(n_1-n_2)}(F^{(n_1-n_2)}(x)) < F^{(n_1-n_2)}(x) + m_2 + m_1 < x + 2(m_2 - m_1)$$

and therefore, inductively,

$$F^{n(n_1-n_2)}(x) < x + n(m_2 - m_1)$$

By the definition of the rotation number, this implies

$$\rho = \lim_{n\to\infty} \frac{F^{n(n_1-n_2)}(x) - x}{n(n_1 - n_2)} < \frac{m_2 - m_1}{n_1 - n_2}$$

which implies the right hand side of (15.4). If $n_1 < n_2$ we just repeat completely analogous calculations to get $\rho > (m_1 - m_2)/(n_2 - n_1)$ which also in this case implies the right hand side of (15.4).

To prove the converse, it is sufficient to show that the negation of the left hand side implies the negation of the right hand side, i.e. that if $F^{n_1}(x) + m_1 \geq F^{n_2}(x) + m_2$ holds for every $x \in \mathbb{R}$ then $n_1\alpha + m_1 \geq n_2\alpha + m_2$. But this follows by exactly the same arguments as above by just inverting the inequalities. □

*Proof of Theorem.* Let $F$ be a lift of $f$ and $F_\rho$ be the lift $F_\rho(x) = x + \rho$ of the rigid rotation $f_\rho$. We will construct a semi-conjugacy $H : \mathbb{R} \to \mathbb{R}$ between these two lifts and show that this semi-conjugacy is periodic and thus "projects" to a seminconjugacy between $f, f_\rho$. We start by using Proposition 15.10 to construct a bijection between two individual orbits. More precisely, for arbitrary $x \in \mathbb{R}$ let

$$\theta_x := \{F^n(x) + m : n, m \in \mathbb{Z}\} \quad \text{and let} \quad \theta_\rho := \{n\rho + m : n, m \in \mathbb{Z}\}.$$

By proposition 15.10 these two sets are ordered in the same way. We define a function $H : \mathbb{R} \to \mathbb{R}$ by

$$H(y) = \sup\{n\rho + m : F^n(x) + m \leq y\}.$$

We remark first of all that $H$ is non-decreasing and constant on each closed interval $[a, b] \subseteq \mathbb{R} \setminus \bar{\theta}_x$ (if such an interval exists). Indeed, since $F^n(x) + m \leq a$ if and only if $F^n(x) + m \leq b$ for any $m, n \in \mathbb{Z}$ and therefore $H(a) = H(b)$. Secondly, notice that $\theta_\rho$ is dense in $\mathbb{R}$, since $\{n\rho \mod 1 : n \in \mathbb{Z}\}$ is exactly the orbit of 0 under the irrational circle rotation $f_\rho$ and we know that this set is dense in $[0, 1]$. Therefore it follows from the definition of $H$ that

$$H(F^n(x) + m) = n\rho + m$$

for every $m, n \in \mathbb{Z}$. Thus $H$ maps the orbit $\theta_x$ of $x$ under $F$ to the orbit $\theta_\rho$ of 0 under $F_\rho$. Since $H$ is monotone and the range of $H$ is dense, it follows that $H$ is continuous and surjective. In particular, since it clearly conjugates two individual orbits, this conjugacy passes to the closure and thus we have that $H$ is a conjugacy between $\overline{\theta}_x$ and $\overline{\theta}_\rho = \mathbb{R}$. Moreover, since $H$ is constant on each interval in the complement of $\overline{\theta}_x$ it is a semiconjugcy on all of $\mathbb{R}$. To see that $H$ is periodic we have

$$\begin{aligned} H(y+1) &= \sup\{n\rho + m : F^n(x) + m \leq y + 1\} \\ &= \sup\{n\rho + m : F^n(x) + m - 1 \leq y\} \\ &= \sup\{n\rho + m - 1 : F^n(x) + m - 1 \leq y\} + 1 \\ &= H(y) + 1. \end{aligned}$$

Therefore, we can define $h(y) = H(y) \bmod 1$, and we obtain a function $h : \mathbb{S}^1 \to \mathbb{S}^1$ which is continuous, non-decreasing and surjective and such that $h \circ f = f_\rho \circ h$.

Finally, if there exists a point $x$ whose orbit is dense for $f$, we carry out the construction described above for that orbit in which case we have that also $\theta_x$ is dense in $\mathbb{R}$. Thus the function $H$ is also injective and it follows that it is a homeomorphism and thus $h$ is also a homeomorphism and $f, f_\rho$ are topologically conjugate.

□

## 15.4 Denjoy counter-example

**Theorem 15.11.** *There exists a circle homeomorphism without periodic points for which not all orbits are dense.*

We just give a sketch of the construction. Start with a rigid circle irrational circle rotation

$$f(x) = x + \theta$$

of a circle $\mathcal{S}$ of total length 1. Choose an arbitrary point $x_0$ and a sequence $\{\ell_k\}_{k=-\infty}^{+\infty}$ of positive numbers with

$$\sum_{k=-\infty}^{+\infty} \ell_k = L < \infty.$$

We now perform a "surgery" on the circle as follows:

> "cut" the circle open at each point $x_n$ of the orbit of $x_0$ and replace the point $x_n$ with an "arc" $I_n$ of length $\ell_n$.

This yields a new circle $\tilde{\mathcal{S}}$ of total length $1 + L$. We can define a map

$$\tilde{f} : \tilde{\mathcal{S}}^1 \to \tilde{\mathcal{S}}^1$$

as follows. First of all let $int(I_n)$ denote the *interior* of $I_n$, i.e. the interval $I_n$ without its endpoints, and define

$$\tilde{\mathcal{C}} = \tilde{\mathcal{S}} \setminus \bigcup_{n=-\infty}^{\infty} int(I_n).$$

There is a natural bijection between the set $\tilde{\mathcal{C}}$ and the set

$$\mathcal{C} = \mathcal{S} \setminus \bigcup_{n=-\infty}^{\infty} x_n$$

since the surgery did not affect any point lying outside the orbit $x_0$. Therefore we define $\tilde{f}$ on $\tilde{\mathcal{C}}$ simply as $f$ on $\mathcal{C}$ using this bijection. It therefore just remains to define $\tilde{f}$ on the union of the intervals $I_n$. To do this, we just define a family of homeomorphisms

$$f_n : I_n \to I_{n+1}$$

for $n \in \mathbb{Z}$ in an essentially arbitrary way. These could be, for example, just linear rescalings depending on the relative lengths of $I_n$ and $I_{n+1}$.

This construction clearly gives a bijection $\tilde{f} : \tilde{\mathcal{S}} \to \tilde{\mathcal{S}}$ and it is clear that $\tilde{f}$ has no periodic points and that all orbits in $\tilde{\mathcal{C}}$ are not dense in $\tilde{\mathcal{S}}$. It is not difficult to show that $\tilde{f}$ is a homeomoprhism and, with additional work, it is also possible to show that $\tilde{f}$ can be constructed to be a $C^1$ diffeomorphism. It is however not possible to construct such a counterexample for $\tilde{f}$ a $C^2$ diffeomorphism.