Checking avoiding sure loss and a problem in gambling

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Avoiding sure loss

- A possibility space $\mathcal{X}$
- A gamble g
- A set of desirable gambles $\mathcal{D}$
- Desirability axioms
- $\mathcal{D}$ avoids sure loss if for all $n \in \mathbb{N}$, $g_i \in \mathcal{D}$ and $\lambda_i \geq 0$:

$$\sup_{x \in \mathcal{X}} \left(\sum_{i=1}^{n} \lambda_i g_i(x) \right) \geq 0. \tag{1}$$

Linear programming problems

We can check whether $\mathcal{D}$ avoids sure loss by solving linear programming problems

$$
\begin{aligned}
&(P) \quad && \min \quad \alpha \\
&\text{s.t.} && \forall x \in \mathcal{X} : \sum_{i=1}^{n} \lambda_i g_i(x) \leq \alpha \\
&\text{where} && \lambda_i \geq 0.
\end{aligned}
$$

$$
\begin{aligned}
&(D) \quad && \forall g_i \in \mathcal{D} : \sum_{x \in \mathcal{X}} g_i(x) p(x) \geq 0 \\
& && \sum_{x \in \mathcal{X}} p(x) = 1 \\
&\text{where} && p(x) \geq 0.
\end{aligned}
$$

Gambling company

Consider a gambling company offering bets

Premier League Winner 16/17

Premier League | Tuesday 16th May 2017 | 16:00

Outrights

Win Outright	Each Way: 1/5 for first 3 places
Manchester City 11/8	Manchester Utd 5/2
Chelsea 9/2	Arsenal 9/1
Liverpool 12/1	Tottenham 20/1
Leicester 66/1	Everton 100/1
West Ham 250/1	Southampton 500/1
Middlesbrough 750/1	Swansea 1000/1
West Brom 1000/1	Stoke 1000/1
Bournemouth 1000/1	Hull 1000/1
Watford 1000/1	Burnley 1000/1
Crystal Palace 1000/1	Sunderland 1500/1

Can we exploit this situation to make a profit?
On the other hand, if we were the betting company, then we would like to prevent gamblers to earn money.

Formulate a problem

- We can view these odds as a set of desirable gambles $\mathcal{D}$ to the company.
- This set avoids sure loss $\Longrightarrow$ the company avoids sure loss.
- Checking whether $\mathcal{D}$ avoids sure loss is very quick.

Theorem 1

Let $\mathcal{X} = \{x_1, \dots, x_n\}$. Suppose a_i/b_i are odds on x_i where a_i and $b_i \geq 0$. For each $i = 1, \dots, n$,

$$g_i(x) := \begin{cases} -a_i & \text{if } x = x_i \\ b_i & \text{otherwise.} \end{cases} \tag{2}$$

Then $\mathcal{D} = \{g_1, \dots, g_n\}$ avoids sure loss if and only if

$$\sum_{i=1}^{n} \frac{b_i}{a_i + b_i} \geq 1. \tag{3}$$

Gambling company

Consider several gambling companies offering bets

Sort By: Favourite																												
Man City	7/5	11/8	11/8	11/8	11/8		11/8	6/4	11/8	7/5	7/5	11/8	7/5	27/20	6/4	11/8	11/8	23/17	7/5	6/4	7/5	23/17	7/5	6/4	7/5	7/5		6/4
Man Utd	11/4	5/2	11/4	11/4	11/4		11/4	5/2	11/4	11/4	13/5	3	3	11/4	11/4	11/4	11/4	5/2	11/4	14/5	11/4	5/2	11/4	3	11/4	14/5		3
Chelsea	9/2	9/2	9/2	5	5		9/2	9/2	9/2	5	9/2	5	9/2	9/2	5	5	9/2	9/2	5	9/2	5	9/2	9/2	5	9/2	23/5		24/5
Arsenal	10	9	10	10	10		10	11	8	11	11	10	10	9	11	10	10	11	11	11	11	11	10	11	9	57/5		57/5
Liverpool	14	12	14	14	14		14	14	10	14	14	11	12	11	14	14	14	12	14	14	14	12	14	14	12	73/5		74/5
Tottenham	20	20	20	20	20		20	20	10	22	22	16	16	18	20	18	20	20	22	20	22	20	20	22	16	21		21
Leicester	66	66	66	66	50		66	40	40	66	66	66	66	50	50	66	66	70	66	80	66	70	70	70	66	70		69
Everton	100	100	80	80	80		66	80	66	80	80	66	100	100	66	80	80	79/2	80	100	80	80	100	100	80	123		129
West Ham	150	250	150	150	150		200	100	100	150	200	200	150	100	66	125	100	175	150	250	150	175	200	200	100	284		198
Southampton	600	500	250	500	500		500	250	200	500	600	200	500	250	250	250	500	600	500	250	500	600	250	500	250	864		495
Middlesbrough	800	750	500	750	900		500	1000	750	750	500	500	500	1000	350	750	500	299/4	750	1000	750	750	500	750	500	759		79
Crystal Palace	1000	1000	750	750	900		1000	500	750	1000	1000	1000	750	750	750	750	1000	125	1000	1000	1000	750	750	1000	1000	949		
Hull	1000	1000	1000	750	1000		750	750	700	750	750	1000	500	1000	500	750	500	500	750	750	750	500	750	1000	750	949		495
Stoke	1000	1000	300	750	1000		500	500	750	1000	1000	750	750	500	750	500	750	36625	1000	1000	1000	1000	500	1000	1000	949		495
Swansea	1000	1000	1000	1000	1500		1000	1000	1000	1000	1000	1000	1000	1000	750	500	750	100	1000	1000	1000	750	750	1000	1000	949		495
Watford	1000	1000	750	1000	1500		750	1000	1000	1000	1000	1500	1000	1000	1000	1000	500	100	1000	1000	1000	1000	750	1000	1000	949		495
West Brom	1500	1000	1000	1000	1250		500	1000	750	1500	1000	1000	1000	1000	1000	1000	1000	125	1500	1000	1500	1250	750	1500	1000	949		495
Bournemouth	1500	1000	1000	1000	1750		750	1000	1000	1500	1000	750	1000	1000	500	1500	1000	125	1500	1000	1500	1250	1000	1500	2000	949		495
Burnley	1000	1000	1000	1000	2000		750	1000	500	1500	1000	1000	1000	1500	500	1000	750	100	1500	1000	1500	1000	500	1000	750	949		495
Sunderland	1500	1500	1000	1000	1750		750	1000	1000	1500	1000	1500	1000	1000	750	1000	1000	669/4	1500	1000	1500	1000	1000	1500	2000	949		495

Theorem 2

Let $\mathcal{X} = \{x_1, ..., x_n\}$. Suppose there are m different companies. For each $k = 1, .., m$, a_{ik}/b_{ik} is betting odds on x_i provided by company k where a_{ik} and $b_{ik} \geq 0$. For each $i = 1, ..., n$ and $k = 1, ..., m$,

$$g_{ik}(x) := \begin{cases} -a_{ik} & \text{if } x = x_i \\ b_{ik} & \text{otherwise.} \end{cases} \tag{4}$$

Let a_i^*/b_i^* be the maximum betting odds on outcome x_i, that is,

$$a_i^*/b_i^* := \max_k \{a_{ik}/b_{ik}\} . \tag{5}$$

Then $\mathcal{D} = \{g_{ik} : i = 1, ..., n, \; k = 1, ..., m\}$ avoids sure loss if and only if

$$\sum_{i=1}^{n} \frac{b_i^*}{a_i^* + b_i^*} \geq 1 \tag{6}$$

Free coupons

- Suppose the company offers an extra bet which is not fit for eq. (2). This bet can be viewed as another desirable gamble f. For example, a "free coupon".
- Check whether the company still avoids sure loss, i.e. whether $\mathcal{D} \cup f$ avoids sure loss.

Discussion 1

$\mathcal{D} \cup f$ avoid sure loss if or all $n \in \mathbb{N}$, $g_i \in \mathcal{D}$,$\lambda_i \geq 0$ and $\alpha \geq 0$:

$$\sup_{x \in \mathcal{X}} \left(\sum_{i=1}^{n} \lambda_i g_i(x) + \alpha f(x) \right) \geq 0. \tag{7}$$

$\overline{E}(f) := \inf \{\alpha \in \mathbb{R} : \alpha - f \geq \sum_{i=1}^{n} \lambda_i g_i, n \in \mathbb{N}, g_i \in \mathcal{D}, \lambda_i \geq 0\}.$

Theorem 3

$\mathcal{D} \cup f$ avoids sure loss if and only if $\overline{E}(f) \geq 0$.

A proof is easy and we can reduce the size of a linear programming problem.

Q: Can we simplify this theorem, e.g. without solving a linear programming problem?

Discussion 2

- Two sets of desirable gambles $\mathcal{D}_1$ and $\mathcal{D}_2$.
- $\mathcal{D}_1$: large, avoiding sure loss (formulate from given odds)
- $\mathcal{D}_2$: small (formulate from free coupons)

Aim: check whether $\mathcal{D}_1 \cup \mathcal{D}_2$ avoid sure loss?

Possible way: add one desirable gamble to $\mathcal{D}_1$ and check whether it avoids sure loss. Do we have a better way?

Discussion 3

Get free coupon if you lose

- You first bet with the company, say Man city 11/8.
- If Man city wins, you get your reward.
- If not, the company give you a free coupon to bet on other tournaments.

Aim: formulate this problem to a problem of checking avoiding sure loss.