DURHAM UNIVERSITY

Department of Mathematical Sciences

Levels 3 and 4 Mathematics modules
Course Booklet
2016 - 2017

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Contents

1 General Information

Welcome to 3H or 4H Mathematics! About 1,200 undergraduates take modules provided by the Department. This booklet provides information on third and fourth-year modules offered by the department.
It also contains summary information on key policies related to assessment and academic progress.

Full details of the department's policies and procedures are available in the departmental degree programme handbooks at http://www.dur.ac.uk/mathematical.sciences/teaching/handbook/ , which also contains on on-line version of the course descriptions contained in this booklet.

Information concerning general University regulations, examination procedures etc., are contained in the Faculty Handbooks (`www.dur.ac.uk/faculty.handbook`) and the University Calendar, which provide the definitive versions of University policy. The Teaching and Learning Handbook (`www.dur.ac.uk/teachingandlearning.handbook`) contains information about assessment procedures, amongst other things.

You should keep this booklet for future reference. For instance, prospective employers might find it of interest. Whether you are entering the final year of your course, or the third year of a four year course, there is a good choice of options and you can study those topics which you find most interesting. You can look forward to an enjoyable year.

1.1 Useful Contacts

The first point of contact for issues referring to a particular course or module should be the relevant lecturer. For more general questions or difficulties you are welcome to consult the Course Director or your Adviser. For queries relating to teaching issues, for example registration, timetable clashes, support for disabilities or illness, you should visit the department to speak to someone in the main Maths Office (CM201), or send an email to maths.teaching@durham.ac.uk.

Head of Department: maths.head@durham.ac.uk
Director of Undergraduate Studies:
Dr Matthias Troffaes (CM212, matthias.troffaes@durham.ac.uk)

The Course Directors for students are determined by their programme and level of study as follows:
Students on Mathematics programmes at level one:
maths.1hcoursedirector@durham.ac.uk
Students on Mathematics programmes at level two:
maths.2hcoursedirector@durham.ac.uk
Students on Mathematics programmes at levels three and four:
maths.34hcoursedirector@durham.ac.uk
Students on Natural Sciences and Combined Honours programmes at all levels:
maths.natscidirector@durham.ac.uk
Students on programmes other than Mathematics and Natural Sciences and Combined Honours at all levels:
maths.otherprogdirector@durham.ac.uk

We may also wish to contact you! Please keep the Mathematics Office informed of your current term-time residential address and e-mail address.

1.2 Registration for 3/4H

You will register for the required number of modules in October. You may attend additional modules during the first few weeks of the Michaelmas Term. If you then decide that you want to change one or more of your modules you must contact maths.teaching@durham.ac.uk or visit the main Maths office (CM201). Any such change should be completed during the first three weeks of the Michaelmas Term.

1.3 Course Information

Term time in Durham is Michaelmas (10 weeks), Epiphany (9 weeks) and Easter (9 weeks). There are 22 *teaching* weeks, and the last seven weeks are dedicated to private revision, examinations and registration for the subsequent academic year. Timetables giving details of places and times of your commitments are available on Departmental web pages and noticeboards in the first floor corridor of the Department. It is assumed that you read them!
You can access your own Maths timetable at `www.maths.dur.ac.uk/teaching/` and then clicking on the 'My Maths timetable' link.

Also, teaching staff often send you important information by e-mail to your local '@durham.ac.uk' address, and so you should scan your mailbox regularly.

1.4 Assessment

Full details of the University procedures for Examinations and Assessment may be found in Section 6 of the Learning and Teaching Handbook, http://www.dur.ac.uk/learningandteaching.handbook/. The Department's policies and procedures are described in the departmental degree programme handbook, http://www.dur.ac.uk/mathematical.sciences/teaching/handbook/ . The Department follows the marking guidelines set out by the University Senate:

Degree Class	Marking Range(%)
I	70 – 100
II(i)	60 – 69
II(ii)	50 - 59
III	40 – 49
Fail	0 – 39

With the exception of final year projects (MATH 3382 Project III and MATH 4072 Project IV), Statistical Methods III (MATH 3051) and Mathematics Teaching (MATH 3121), assessment for third and fourth year modules is by written examination. For Statistical Methods III (MATH 3051), 30% of the module is based on in-year exams, one at the end of the Michaelmas term and one at the end of the Epiphany term. For Mathematics Teaching (MATH 3121), 20% of the assessment is based on the school file, 30% on summative coursework, 10% on a presentation, and 40% is based on a written report of the students' placement projects. For Project III (MATH 3382) and Project IV (MATH 4072), 15% of the assessment is based on a presentation and poster, and 85% is based on a written report. All courses include either summative or formative assessed work, with assignments being set on a regular basis in lecture-based courses. The purpose of formative and summative assessment of coursework is to provide feedback to you on your progress and to encourage effort all year long.

Regular assignments are marked A-E to the following conventions:

Grade	Equivalent Mark	Quality
A	$\geq$ 80%	*Essentially complete and correct work*
B	60%—79%	*Shows understanding, but contains a small number of errors or gaps*
C	40%—59%	*Clear evidence of a serious attempt at the work, showing some understanding, but with important gaps*
D	20%—39%	*Scrappy work, bare evidence of understanding or significant work omitted*
E	<20%	*No understanding or little real attempt made*

Use of Calculators in Exams The use of electronic calculators is allowed in some module examinations and other module assessments. Each student taking modules offered by departments or schools within the science faculty, which specify that calculators be allowed in assessments, will be offered a calculator, free of charge, at the beginning of their course. The model will be a Casio fx-83 GTPLUS or a Casio fx-85 GTPLUS.

Calculators will become the property of students who will be responsible for their upkeep. No replacement calculators will be provided free of charge, but may be available to purchase from departments/schools, depending on availability. The specified calculator will also be generally available, in shops and online, should a replacement purchase be necessary.

Where the use of calculators is allowed in assessments, including examinations, the only models that will be allowed are either a Casio fx-83 GTPLUS or a Casio fx-85 GTPLUS. In particular, examination invigilators will refuse to allow a candidate to use any calculator other than the model(s) specified, which will be explicitly stated on the front of the examination paper. During examinations no sharing of calculators between candidates will be permitted, nor will calculators or replacement batteries be supplied by the Department or the Student Registry Office.

1.5 Academic progress

The Department is responsible for ensuring that students are coping with the courses and meeting their academic commitments.
For 3rd and 4th year modules you are required:
- for Project modules, to attend meetings arranged with the supervisor, submit your poster and draft and final written reports on time, and give a presentation as scheduled.
- to submit summative or formative assessed work on time to a satisfactory standard.
Assessed work which is graded D or E is counted as being of an unsatisfactory standard.
Attendance and submission of work is monitored through a database. It is your responsibility to ensure that your attendance is recorded by signing the relevant attendance sheets.
Students who are not keeping up with their commitments will be contacted by course directors to help get them back on track.
Persistent default will result in a formal written warning, which may be followed by the initiation of Faculty procedures.

Full details of academic progress requirements for the department are available in the departmental degree programme handbook, http://www.dur.ac.uk/mathematical.sciences/teaching/handbook/ .

1.6 Durham University Mathematical Society

MathSoc: Necessary and Sufficient

Durham University Mathematical Society, affectionately known as MathSoc, provides an opportunity for maths students (or anyone with an interest in maths) to meet away from lectures.

We arrange a variety of events throughout the year, including bar crawls, talks by guest speakers, a Christmas meal, and the highlight of the year – a trip to see Countdown being filmed! So there's something for everyone. We are currently sponsored by EY so we are able to offer many exclusive networking opportunities and careers events!

MathSoc works with the Maths Department to arrange Undergraduate Colloquia, where departmental and external lecturers give relaxed, inspiring talks on their current research. These cover a wide range of mathematical topics with previous titles including 'Dot-dots, zig-zags and plank-planks' and 'Defects of integrable field theory'. These are at a level such that anyone with an interest in maths can enjoy them and they aim to motivate an interest in an area of maths you may not have seen before.

We have our own website (durhammathsoc.wix.com/mathematicalsociety), where you will find all the most up-to-date information about the society. Here you will also find our second-hand book list, which has many of the books needed for courses for much cheaper than you will find them in the shops. Last year people saved up to £50 by using this service!

If you would like any more information about either the society itself, or advice on any other aspect of the maths course, please do not hesitate to get in touch with any of our friendly exec members listed below or via the society email address (mathematical.society@durham.ac.uk). You can also find us on Facebook by searching for "The Official Durham MathSoc Group".

To join:

Come and see our stand at the freshers' fair, or email us at any time: it costs only £7 for life membership, or £4 for a year. You can sign up on the Durham Students Union Website (http://www.durhamsu.com/groups/mathematical). You can also find us on Facebook by searching for "The Official Durham MathSoc Group" and follow @DUMathsoc on Twitter!

1.7 Disclaimer

The information in this booklet is correct at the time of going to press in May 2016. The University, however, reserves the right to make changes without notice to regulations, programmes and syllabuses. The most up-to-date details of all undergraduate modules can be found in the Faculty Handbook on-line at `www.dur.ac.uk/faculty.handbook/`.

1.8 Booklists and Descriptions of Courses

The following pages contain brief descriptions of the level 3 and 4 modules available to you.These supplement the official descriptions in the module outlines in the faculty handbook which can be found at
`http://www.dur.ac.uk/faculty.handbook/module_search/?search_dept=MATH&search_level=3` or
`http://www.dur.ac.uk/faculty.handbook/module_search/?search_dept=MATH&search_level=4`. Note that the official module outlines contain information on module pre- and co-requisites, excluded combinations, assessment methods and learning outcomes. The descriptions which follow supplement this by providing a list of recommended books and a brief syllabus for each module.

For some modules you are advised to buy a particular book, indicated by an asterisk; for others a choice of titles is offered or no specific recommendation is given. There are also suggestions for preliminary reading and some time spent on this during the summer vacation may well pay dividends in the following years.

Syllabuses, timetables, handbooks, exam information, and much more may be found at `www.maths.dur.ac.uk/teaching/`, or by following the link ‘teaching’ from the Department’s home page (`www.maths.dur.ac.uk`). These syllabuses are intended as guides to the modules. The definitive information on course content and expected learning outcomes is in the official module outlines.

1.8.1 CRYPTOGRAPHY AND CODES III – MATH3401 – (38 lectures)

Dr S. Darwin / Dr P. Vishe

This course deals with two very concrete applications in data transfer which heavily involve very abstract mathematics. The first one is about error-correcting codes. In particular they are used in compact discs to cope with scratches. But error-correcting codes are also used more widely, e.g., for data transmission over noisy channels.

The next application is cryptography. Nowadays it is not only used by spies and secret agents. It makes internet browsing and in particular internet banking secure. Cryptography also prevents unauthorized people from capturing and listening to your mobile phone talks.

1st term. We start with the basics of error-correcting codes, working with vectors and matrices with coefficients in $\mathbb{Z}_p$ (p prime) or more generally over a general finite field, and discuss the error-detection and error-correction capabilities of various codes, such as the Golay code. In particular, we also study the special error-correcting capabilities of the Reed-Solomon code used on CD's.

2nd Term. We start with the notion of trapdoor function which is in the essence of the whole area of open key cryptography. We consider several number theoretical problems (i.e. factorization and discrete logarithm problem) which provide us with examples of trapdoor functions. Then we go through Diffie-Hellman key exchange scheme and RSA cryptosystem. We end up with elliptic curves and the modification of Diffie-Hellman based on it.

Material from Linear Algebra I and Elementary Number theory II will be heavily used in this course. Algebra II may also help.

Recommended Books

The first part of the course is based on **'A First Course in Coding Theory'** by Raymond Hill (Oxford University Press, ISBN 0198538030, about £36), but **'Coding theory: a first course'** by San Ling and Chaoping Xing (Cambridge University Press, ISBN 0521529239, about £33) also has large parts in common with it.

Oliver Pretzel **'Error-Correcting Codes and Finite Fields'** Oxford University Press, ISBN 0192690671. It is more substantial than the first book.

W. Trappe and L. Washington, **Introduction to Cryptography with Coding Theory**, Prentice Hall.

L. Washington, **Elliptic Curves: Number Theory and Cryptography**, Chapman and Hall/CRC.

Term 1

Introduction to codes: Hamming distance, Error-detection and -correction: procedures, capabilities, probabilities. Equivalence of codes.

Revision of linear algebra over $\mathbb{F}_p$.

Linear codes: rank, generator and check matrices. Equivalence. Dual codes. Decoding methods: standard array, syndromes.

Hamming codes, sphere-packing bound and perfect codes. Decoding with a Hamming code. Golay Codes.

Finite fields $\mathbb{F}_{p^r}$. Linear codes over such fields. Cyclic codes, BCH codes, Reed-Solomon codes.

Burst-errors and ways to fix them: interleaving, p-ary codes made from p^r-ary codes.

Term 2

Introduction to open-key cryptography, the notion of 'trapdoor function'. The factorization and the discrete logarithm problems.

Diffie-Hellman key exchange scheme.

RSA cryptosystem and possible attacks on it. Various factorization methods and conditions on the parameters on RSA.

Elliptic curves over rational numbers and over finite fields.

Elliptic curve Diffie-Hellman scheme.

Lenstra factoring algorithm.

1.8.2 DECISION THEORY III – MATH3071 (38 lectures)

Dr C. Caiado / Prof F. Coolen

Decision theory concerns problems where we have a choice of decision, and the outcome of our decision is uncertain (which describes most problems!). Topics for the course typically include the following (contents may vary a bit from year to year):

(1) Introduction to the ideas of decision analysis.

(2) Decision trees - how to draw and how to solve.

(3) Representing decision problems using influence diagrams.

(4) Quantifying rewards as utilities - informal ideas, formal construction, relevance to statistical analysis and multi-attribute utility. Von Neumann-Morgenstern theory of maximum expected utility.

(5) Alternative decision criteria and reflection on normative versus descriptive theories (including prospect theory).

(6) Applications, e.g. in health, industry and insurance.

(7) Bargaining problems: Nash theory for collaborative games.

(8) Group decisions and social choice: Arrow's theory on social welfare functions, Harsanyi's utilitarianism, further developments.

(9) Game theory: two-person zero-sum games, brief discussion of more complex games and applications in e.g. biology.

(10) Further topics related to recent developments in research and applications.

Recommended Books

M. Peterson, **An Introduction to Decision Theory**, Cambridge University Press, 2009.
J.Q. Smith, **Decision Analysis - A Bayesian Approach**, Chapman & Hall, 1988.
D.V. Lindley, **Making Decisions**, (2nd edition), Wile, 1985.
S. French, **Decision Theory: An Introduction to the Mathematics of Rationality**, Ellis Horwood, Chichester, 1986.
M.H. DeGroot, **Optimal Statistical Decisions**, McGraw-Hill.
R.T. Clemen, **Making Hard Decisions**, (2nd edition), Duxbury Press, 1995.

Preliminary Reading

While any of the recommended books would make a good preparation for the course (except De Groot, which is rather too technical) you are advised to look at **Making Decisions** by Lindley, which gives an interesting and fairly painless introduction to some of the most important ideas. Peterson is relatively cheap and covers a substantial part of the topics in the course, however it has slightly more emphasis on philosophical aspects and less on mathematics than our presentation.

Aim: To describe the basic ingredients of decision theory, for individuals and for groups, and to apply the theory to a variety of interesting and important problems.

Term 1 (20 lectures)

Introduction to Decision Analysis: Decision trees, sequential decisions; uncertainties and values, solution by backward induction; perfect information and cost of information; representation by influence diagrams.

Utility: Von Neumann - Morgenstern theory of maximum expected utility; utility of money and risk aversion; multi-attribute utility; relevance to statistical analysis.

Alternative theories: Alternative decision criteria, prospect theory.

Applications: Some examples of applications in e.g. health, industry and insurance.

Term 2 (18 lectures)

Bargaining: Nash theory for collaborative games; alternative theories.

Group Decisions and Social Choice: Arrow's theory on social welfare functions; Harsanyi's theory on utilitarianism; alternative theories and recent developments.

Game Theory: Two-person zero-sum games; brief discussion of non-constant sum games and other more complex games, and of applications in e.g. biology.

Further topics: Selection of topics related to recent developments in research and applications.

1.8.3 DIFFERENTIAL GEOMETRY III – MATH3021 (38 lectures)

Dr P. Tumarkin

Differential geometry is the study of curvature. Historically it arose from the application of the differential calculus to the study of curves and surfaces in 3-dimensional Euclidean space. Today it is an area of very active research mainly concerned with the higher-dimensional analogues of curves and surfaces which are known as n-dimensional differentiable manifolds, although there has been a great revival of interest in surfaces in recent years. Differential geometry has been strongly influenced by a wide variety of ideas from mathematics and the physical sciences. For instance, the surface formed by a soap film spanning a wire loop is an example of a minimal surface (that is, a surface whose mean curvature is zero) but the ideas and techniques involved in analysing and characterising such surfaces arose from the calculus of variations and from Riemann's attempt to understand complex analysis geometrically. The interplay of ideas from different branches of mathematics and the way in which it can be used to describe the physical world (as in the case of the theory of relativity) are just two features which make differential geometry so interesting.

In order to keep the treatment as elementary and intuitive as possible this level III course will be almost entirely devoted to the differential geometry of curves and surfaces, although most of the material readily extends to higher dimensions. The techniques used are a mixture of calculus, linear algebra, and topology, with perhaps a little material from complex analysis and differential equations. The course will follow the notes of Woodward and Bolton, and the book by Do Carmo is also be very suitable. You are recommended to buy either of these. Spivak's books are interesting, well written and useful for reference. If you like the format of the Schaum series, then you may find Lipschutz's book helpful.

Recommended Books

M. do Carmo, **Differential Geometry of Curves and Surfaces**, Prentice-Hall.
S. Lipschutz, **Differential Geometry**, Schaums Outline Series, McGraw-Hill.
M. Spivak, **Differential Geometry**, Vols. II & III, Publish or Perish.
* L.M. Woodward and J. Bolton, **Differential Geometry** (Preliminary Version). Duplicated copies of this are available for students from the Departmental Office.

Preliminary Reading

Try looking at the first two chapters of do Carmo or Woodward and Bolton, or Chapters 1-3 of Spivak II. If you cannot get hold of these, then any book which deals with the differential geometry of curves and surfaces in Euclidean 3-space should be useful.

Aim: To provide a basic introduction to the theory of curves and surfaces, mostly in 3-dimensional Euclidean space. The essence of the module is the understanding of differential geometric ideas using a selection of carefully chosen interesting examples.

Term 1 (20 lectures)

Curves: Plane curves. Arc length, unit tangent and normal vectors, signed curvature, Fundamental theorem. Involutes and evolutes. Gauss map, global properties. Space curves. Serret-Frenet formulae. Fundamental theorem. [Global properties]

Surfaces in $\mathbb{R}^n$: Brief review of functions of several variables including differential (with geometric interpretation) and Inverse Function Theorem. Definition of regular surface in and Coordinate recognition lemma. [Change of coordinates.] Curves on a surface, tangent planes to a surface in $\mathbb{R}^n$.

First Fundamental Form: Metric, length, angle, area. Orthogonal and isothermal coordinates. [Orthogonal families of curves on a surface.] [Some more abstract notions such as 'metrics' on open subsets of $\mathbb{R}^n$ can be introduced here.]

Mappings of Surfaces: Definitions, differential, expressions in terms of local coordinates. [Conformal mappings and local isometrics. Examples (conformal diffeomorphisms and isometries of $\mathbb{R}^2$, S^2 and helicoid; conformal diffeomorphism of $S^1(a) \times S^1(b) \subseteq \mathbb{R}^4$ onto torus of revolution in $\mathbb{R}^3$; $\mathbb{R}P^2$ as Veronese surface in $\mathbb{R}^5$)].

Term 2 (18 lectures)

Geometry of the Gauss Map: Gauss map and Weingarten map for surfaces in $\mathbb{R}^3$, second fundamental form, normal (and geodesic) curvature, principal curvatures and directions. Gaussian curvature K and mean curvature H. Second fundamental form as second order approximation to the surface at a point. Explicit calculations of the above in local coordinates. Umbilics. Compact surface in $\mathbb{R}^3$ has an elliptic point.

Intrinsic Metric Properties: Christoffel symbols, Theorema Egregium of Gauss. Expression of K in terms of the first fundamental form. Examples. [Intrinsic descriptions of K using arc length or area]. Mention of Bonnet's Theorem. [Surfaces of constant curvature].

Geodesics: Definition and different characterisations of geodesics. Geodesics on surfaces of revolution. Geodesic curvature with description in terms of local coordinates.

Minimal Surfaces: Definition and different characterisations of minimal surfaces. Conjugate minimal surfaces and the associated family. [Weierstrass representation.]

Gauss-Bonnet Theorem: Gauss-Bonnet Theorem for a triangle. Angular defect. Relationship between curvature and geometry. Global Gauss-Bonnet Theorem. Corollaries of Gauss-Bonnet Theorem. [The Euler-Poincaré-Hopf Theorem.]

1.8.4 DYNAMICAL SYSTEMS III – MATH3091 (38 lectures)

TBA / Dr B. van Rees

Dynamical systems is the mathematical study of systems which evolve in time. One classical example is Newtonian dynamics, but the applicability is in fact much wider than this.

This course mainly deals with systems described by (coupled) ordinary differential equations. We start by studying the local behaviour or solutions, both in time and in the neighbourhoods of fixed points. Although one might think that such systems could be solved in closed form with sufficient effort, Poincaré realised a century ago that there are fundamental obstacles to global exact solvability and that most dynamical systems are in fact unsolvable in closed form for all time. This leads us to develop methods to study the global qualitative behaviour of such systems, finishing with a brief introduction to chaotic dynamical systems.

Recommended Books

M.W. Hirsch, S. Smale, R.L. Devaney **Differential Equations, Dynamical Systems, and an Introduction to Chaos**, Academic Press 2003, ISBN 0123497035

This book is very accessible, it is written in a rather informal style and covers lots of examples. It however misses some of the material covered in the Epiphany term.

L. Barreira, C. Valls Ordinary Differential Equations: Qualitative Theory, American Mathematical Soc. 2012, ISBN 082188994X

This book is rather formal and in particular contains proofs of all the essential theorems. Highly recommended for the serious student.

There are many other textbooks on Dynamical Systems. For example:

D.K. Arrowsmith and C.M. Place, **Dynamical Systems**, Chapman & Hall 1992, ISBN 0412390809

P.G. Drazin, **Nonlinear Systems**, CUP 1992, ISBN 0521406684

P. Glendinning, **Stability, Instability and Chaos**, CUP 1994, ISBN 0521425662

F. Verhulst, **Nonlinear Differential Equations and Dynamical Systems**, Springer (2nd Edition) 1996, ISBN 3540609342

Aim: To provide an introduction to modern analytical methods for nonlinear ordinary differential equations in real variables.

Term 1 (20 lectures)

Introduction: Smooth direction fields in phase space. Existence and Uniqueness Theorem and initial-value dependence of trajectories. Orbits. Phase portraits. Equilibrium and periodic solutions. Orbital derivative, first integrals.

Linear Autonomous Systems: Classification of Linear Systems in two and higher dimensions. The exponential map.

Nonlinear systems near equilibrium Hyperbolic fixed points, Stable and Unstable Manifolds. Stable-Manifold Theorem, Hartman-Grobman Theorem.

Stability of fixed points Definitions of stability, Orbital derivative, first integrals, Lyapounov functions and Stability Theorems.

Term 2 (18 lectures)

Local Bifurcations: Classification for 1d systems, Some 2d examples, Robustness of bifurcations, Example of a global bifurcation

Orbits and Limit Sets: Omega-limit sets, Poincare-Bendixson Theorem, Index theorem, Absorbing sets and limit cycles

Lorenz system and Introduction to Chaos

1.8.5 GALOIS THEORY III – MATH3041 (38 lectures)

Prof. V. Abrashkin

The origin of Galois Theory lies in attempts to find a formula expressing the roots of a polynomial equation in terms of its coefficients. Evarist Galois was the first mathematician to investigate successfully whether the roots of a given equation could in fact be so expressed by a formula using only addition, subtraction, multiplication, division, and extraction of n-th roots. He solved this problem by reducing it to an equivalent question in group theory which could be answered in a number of interesting cases. In particular he proved that, whereas all equations of degree 2, 3 or 4 could be solved in this way, the general equation of degree 5 or more could not. Galois Theory involves the study of general extensions of fields and a certain amount of group theory. Its basic idea is to study the group of all automorphisms of a field extension. The theory has applications not only to the solution of equations but also to geometrical constructions and to Number Theory.

Recommended Books

*I. Stewart, **Galois Theory**, Chapman and Hall, ISBN 0412345404; £23.99
E. Artin, **Galois Theory**, Dover, ISBN0486623424; about £6
J. Rotman, **Galois Theory**, Springer
M.H. Fenrick, **Introduction to the Galois Correspondence**, Birkhaser
D.J.H. Garling, **A Course in Galois Theory**, Cambridge University Press
P.J. McCarthy, **Algebraic Extensions of Fields**, Chelsea
H.M. Edwards, **Galois Theory**, Springer
B.L. Van der Waerden, **Modern Algebra**, Vol. 1, Ungar.

Aim: To introduce the way in which the Galois group acts on the field extension generated by the roots of a polynomial, and to apply this to some classical ruler-and-compass problems as well as elucidating the structure of the field extension.

Term 1 (20 lectures)

Introduction: Solving of algebraic equation of degree 3 and 4. Methods of Galois theory.

Background: Rings, ideals, a ring of polynomials, fields, prime subfields, factorisation of polynomials, tests for irreducibility.

Field Extensions: Algebraic and transcendental extensions, splitting field for a polynomial, normality, separability.

Fundamental Theorem of Galois Theory: Statement of principal results, simplest properties and examples.

Term 2 (18 lectures)

Galois Extensions with Simplest Galois Groups: Cyclotomic extensions, cyclic extensions and Kummer Theory, radical extensions and solvability of polynomial equations in radicals.

General Polynomial Equations: Symmetric functions, general polynomial equation and its Galois group, solution of general cubic and quartic, Galois groups of polynomials of degrees 3 and 4, non-solvability in radicals of equations of degree ≥ 5.

Finite Fields: Classification and Galois properties, Artin-Schreier Theory, construction of irreducible polynomials with coefficients in finite fields.

1.8.6 GEOMETRY III AND IV – MATH3201/MATH4141 (38 lectures)

Dr A. Felikson

The course deals with the relationship between group theory and geometry as outlined by Felix Klein in the late 19C in his famous 'Erlanger Program'. This will be discussed through different types of geometries in 2-dimensions, namely Euclidean, affine, projective, Möbius and hyperbolic geometry.

Each of these geometries may be characterized as the study of properties of figures which are preserved by a certain group of transformations. For example, Euclidean geometry of the plane is concerned with properties which depend on the notion of distance. The corresponding group of distance preserving transformations is the Euclidean group consisting of translations, rotations, reflections and glide reflections. The affine group, which contains the Euclidean group, is the group of transformations of the plane which send line to lines and the corresponding geometry, which is more general than Euclidean geometry, is called affine geometry. Projective geometry, which is a further generalization, provides a powerful tool for describing many aspects of the geometry of the plane particularly those concerned with the ellipse, parabola and hyperbola. Möbius geometry is the geometry of lines and circles in the plane and the Möbius group is the group of Möbius transformations of the extended complex plane familiar from complex analysis.

All the classical geometries mentioned above play an important role in the construction of hyperbolic geometry. For more than two thousand years, geometers tried to decide whether or not Euclid's parallel postulate is a consequence of his other four postulates. In 1830 hyperbolic geometry was developed independently by Bolyai, Gauss and Lobatchevsky, and so settled this problem. Since its discovery, the hyperbolic geometry has had a profound effect on the foundations of geometry, recent development of topology, and on various aspects of mathematical physics, especially relativity theory.

Recommended Books

R. Artzy, **Linear Geometry**, Addison-Wesley, Useful for linear aspects.
H.S.M. Coxeter, **Introduction to Geometry**, Wiley, published 1963. A superb introduction to geometry and full of good things. Many ideas from the course are presented in an elementary way.
P.M. Neumann, G.A. Storey and E.C. Thompson, **Groups and Geometry**. Vol.II is very useful for several parts of the course, particularly Möbius geometry.
E. Rees, **Notes on Geometry**, Springer. An excellent set of notes containing a good deal but not all of the material for the course.
J.W. Anderson: **Hyperbolic Geometry**, Springer Undergraduate Mathematics Series, 1999.
A nice course on Hyperbolic geometry.

Preliminary Reading

A.D Gardiner and C.J. Bradley **Plane Euclidean Geometry**, UKMT, Leeds 2012. A collection of all essentials on Euclidean geometry.

4H reading material: to be announced.

Outline of course **Geometry III/IV**

Aim: To give students a basic grounding in various aspects of plane geometry. In particular, to elucidate different types of plane geometries and to show how these may be handled from a group theoretic viewpoint.

<u>Term 1</u> (20 lectures)
Euclidean Geometry: Basics of Euclidean geometry. Isometries, their linear part and group of isometries, reflections as generators of the group, fixed points. Conjugacy classes. Discrete subgroups of the isometry group. Fundamental domains. Torus and Klein bottle as quotients of $\mathbb{R}^2$ by discrete subgroups which act freely. Klein's Erlanger Program and the relationship between group theory and geometry.

Spherical geometry: Spherical lines, triangle inequality, polars. Congruence of spherical triangles, area of spherical triangles, law of sines and law of cosines. $SO(3)$ as the group of orientation preserving isometries, reflections as generators of the whole isometry group.

Affine and projective geometries: Projective line and projective plane, cross- ratio, homogeneous coordinates, projective transformations as compositions of projections from a point. Affine subgroup of the projective group.

Hyperbolic geometry: Model in two-sheet hyperboloid, $PO(2,1)$ as the isometry group, reflections as generators. Hyperbolic lines, half-planes. Projection to Poincaré disc.

Möbius Transformations: Möbius transformations. Triple transitivity. Conformal transformations are Möbius. Conjugacy classes. Inversion, the inversive group. Similarities of $\mathbb{R}^2$ and isometries of S^2 as elements of the inversive group. Circle-preserving transformations of S^2 are inversive. Orthogonal circles. Cross-ratio. Characterisation of a circle through three points in terms of cross-ratio.

<u>Term 2</u> (18 lectures) **Poincaré models**: Upper half-plane: hyperbolic lines, $PSL(2;\mathbb{R})$ as the group of orientation preserving isometries. Poincaré disc: lines, angles, $PU(1,1)$ as the group of orientation preserving isometries.

Elementary hyperbolic geometry: Euclid's axioms, parallel postulate, parallel lines in the hyperbolic plane, angle of parallelism. Hyperbolic triangles, sum of angles, congruence of triangles, hyperbolic trigonometry, law of sines, law of cosines, area of hyperbolic triangles.

Isometries: classification, conjugacy classes, invariant sets (circles, horocycles and equidistant curves).

Klein disc model: lines, angles, horocycles.

Reflection groups: reflection groups on the sphere, Euclidean plane and hyperbolic plane: fundamental domains, Gram matrices, classification.

Further topics: Hyperbolic surfaces: brief treatment (by example) of surfaces of genus greater than 1 as quotients of the hyperbolic plane by discrete subgroups which act freely, pants decompositions. Hyperbolic space.

1.8.7 MATHEMATICAL BIOLOGY III – MATH3171 (38 lectures)

TBA

Mathematical Biology is one of the most rapidly growing and exciting areas of applied mathematics. Over the past decade research in Biological sciences has evolved to a point that experimentalists are seeking the help of applied mathematicians to gain quantitative understanding of their data. Mathematicians can help biologists to understand very complex problems by developing models for biological situations and then providing a suitable solution. However, for a model to be realistic cross-talk between these two disciplines is absolutely essential.

Mathematics is now being applied in a wide array of biological and medical contexts and professionals in this field are reaping the benefits of research in these disciplines. Examples range from modelling physiological situations e.g. ECG readings of the heart, MRI brain scans, blood flow through arteries, tumor invasion and others. At the cellular level we can ask quantitative questions about the process by which DNA gets "transcribed" (copied) to RNA and then "translated" to proteins, the central dogma of molecular biology. Mathematical Biology also encompasses other interesting phenomena observed in nature, e.g. swimming behavior of microorganisms, spread of infectious diseases, and emergence of patterns in nature. In this course we shall examine some fundamental biological problems and see how to go about developing mathematical models that describe the biological situation with definite predictions that can then be tested to validate the model.

Recommended Books

J.D. Murray, **Mathematical Biology I. An Introduction**, Springer, ISBN 0387952233; (about £20)
J.D. Murray, **Mathematical Biology II. Spatial models and biomedical applications**, Springer, ISBN 0387952284; (about £57)
N. F. Britton, **Essential Mathematical Biology**, Springer, ISBN 185233536X; (about £ 30)
Lee A. Segel **Modelling dynamic phenomena in molecular and cellular biology**, Cambridge University Press, ISBN 052127477X; about £20
J. Keener and J. Sneyd **Mathematical Physiology**, Springer, ISBN 0387983613; (about £50)
L. A. Segel, and L. Edelstein-Keshet **A primer of Mathematical Models in Biology**, Cambridge University Press, ISBN 9781611972498 (about £50)
K. Sneppen, **Models of Life: Dynamics and Regulation in Biological Systems**, Cambridge University Press, ISBN 978-1-107-06910-3 (about £ 60)

None of these books covers the course entirely. However, the course is covered by all. Segel's book is easiest to read followed by Britton. Murray's are excellent. Keener and Sneyd is a very good account of more medical applications of mathematics.

Aim: Study of non-linear differential equations in biological models, building on level 1 and 2 Mathematics.

<u>Term 1</u> (20 lectures)

Introduction to the Ideas of Applying Mathematics to Biological Problems

Reaction Diffusion Equations and their Applications in Biology:

Diffusion of insects and other species. Hyperbolic models of insect dispersal and migration of a school of fish. Modelling the life cycle of the cellular slime mold Dictyostelium discoideum, and the phenomenon of chemotaxis. Glia aggregation in the human brain and possible connection with Alzheimer's disease.

<u>Term 2</u> (18 lectures)

ODE Models in Biology: Enzyme kinetics and the chemostat for bacteria production

The Formation of Patterns in Nature: Pattern formation mechanisms, morphogenesis. Questions such as how does a Diffusion driven instability and pattern formation (Turing instability)

Epidemic Models and the Spread of Infectious Diseases: Epidemic models. Spread of infectious diseases. Simple ODE model. Spatial spread of diseases.

Epidermal and Dermal wound healing • Basic models, Derivation of equations, Stability analysis and travelling wave solutions.

1.8.8 MATHEMATICAL FINANCE III AND IV – MATH3301/4181 (38 lectures)

Dr H. Sayit

Financial mathematics uses mathematical methods to solve problems in financial industry. It draws on tools from probability theory, statistics, partial differential equations, and scientific computing, such as numerical methods, Monte Carlo simulation and optimization. The methods and tools in financial mathematics are widely used in investment banks, commercial banks, hedge funds, insurance companies, corporate treasuries, and regulatory agencies to solve such problems as derivative pricing, portfolio selection, risk management, and scenario simulation. It is one of the fastest developing areas of mathematics and it has brought efficiency and rigor to financial markets over the years. As such it is becoming increasingly important to the financial firms.

This is an introductory course and it covers important tools and ideas of financial mathematics. The first part of this course discusses discrete time financial models. Main concepts of financial mathematics such as forward and futures contracts, call and put options, completeness, self-financing condition, replicating strategies, and risk-neutral measures will be introduced. Pricing problems of European and American options under Cox-Ross-Rubinstein (CRR) Binomial model will be discussed. The Black-Scholes formula for the call price will be derived as a limit of the CRR model. Then the concepts of preferences, utility, and risk aversion will be introduced and the capital-asset pricing model will be discussed in detail.

The second part of the course will discuss continuous time financial models. A short introduction to the Ito's integral and ito's formula will be given. Pricing problems of path dependent options under the standard Black-Scholes model will be discussed. Then short rate models will be introduced and affine term structure of bond prices will be discussed. A short introduction to the Heath-Jarrow-Morton forward rate model will also be given.

Recommended Books

C. Marek and Z. Tomasz, **Mathematics for Finance**, Springer, 2005.
K. Douglas, **Stochastic Financial Models**, Taylor and Francis Group, 2010.
A. Etheridge, **A Course in Financial Calculus**. Cambridge University Press, 2002.
N.H.Bingham and R.Kiesel, **Risk Neutral Valuation**, Springer 1998.
M.Baxter and A. Rennie, **Financial Calculus**, Cambridge University Press, 1996.
T. Bjork, **Arbitrage in Continuous Time**, Oxford University Press, 1999.
E. Shreve, **Stochastic Calculus for Finance (Volumes I and II)**, Springer Finance, 2004.
R.J.Elliot and P.E.Kopp, **Mathematics of Financial Markets**, Springer, 1999.
J.C. Hull, **Options, Futures and Other Derivatives**, Pearson Intern'l Ed., ISBN 978 0136015864.
P. Wilmott, S. Howison and J. Dewynne, **The mathematics of financial derivatives: a student introduction**, CUP, ISBN 0521497892.

Preliminary Reading

Sections 1.3, 1.4, 1.5, 2.10, 4.1, 4.4 of J.C. Hull, **Options, Futures and Other Derivatives**, Pearson Intern'l Ed., ISBN 978 0136015864.

4H reading material references

Will be assigned later.

Aim: To provide an introduction to the mathematical modelling of financial derivative products.

<u>**Term 1**</u> (20 lectures)

Introduction to options and markets. Interest rates and present value analysis, asset price random walks, pricing contracts via arbitrage, risk neutral probabilities.

The Black-Scholes formula. The Black-Scholes formula for the geometric Brownian price model and its derivation.

More general models. Limitations of arbitrage pricing, volatility, pricing exotic options by tree methods and by Monte Carlo simulation.

<u>**Term 2**</u> (18 lectures)

Introduction to stochastic calculus.

The Black-Scholes model revisited. The Black-Scholes partial differential equation. The Black-Scholes model for American options.

Finite difference methods.

1.8.9 NUMBER THEORY III and IV– MATH3031/4211 (38 lectures)

Dr P. Vishe / Dr H. Gangl

With the resolution of Fermat's Last Theorem by Andrew Wiles, the study of Diophantine Equations (equations where the unknowns are to be integers) has come again to the forefront of mathematics. In this course we study some of the techniques, mainly based on ideas of factorization and congruence, which have been applied to the Fermat and other Diophantine equations. We find, for example, the number of solutions in positive integers to

$x^3 + y^3 = z^3$ (the case n=3 of the Fermat equation)
$x^2 = y^3 - 2$ (a case of Mordell's equation)

(The answers are 0 and 1 respectively.)

Our methods lead us to study number systems (rings) other than that of the ordinary integers (e.g. the Gaussian integers), and we consider what we can do when things do not go according to plan, that is, when unique factorization fails. E.g. recall that in the ring $\mathbb{Z}[\sqrt{-5}]$ we have $6 = 2 \times 3 = (1+\sqrt{-5}) \times (1-\sqrt{-5})$. When uniqueness of factorization breaks down in this way, we are led to consider ideals as "ideal numbers" (this is the reason why ideals are called ideals). We use this idea to regain a form of uniqueness of factorization.

We introduce the ideal class group which measures by how much unique factorization fails in number rings. Using a pretty application of the 'Geometry of Numbers', we show that there are only finitely many ideal classes. This leads to the solution of more sorts of diophantine equations. We also study the units of these number rings. These are given by the Dirichlet unit theorem which can be viewed as a generalization of the characterization of the solutions to Pell's equation $x^2 - Dy^2 = 1$.

We finally turn to some analytic tools and study the zeta function of quadratic fields, which generalize the famous Riemann zeta function. In particular, we discuss the (analytic) class number formula.

Prerequisites

MATH2581 Algebra II. Elementary Number Theory and Cryptography II (MATH2591) and Galois Theory III (MATH3041) are helpful on occasion but not required.

Recommended Books

*I.N. Stewart and D.O. Tall, **Algebraic Number Theory and Fermat's Last Theorem**, A.K. Peters, 2001, ISBN 1568811195
A. Baker, **A Concise Introduction to the Theory of Numbers**
P. Samuel, **The Algebraic Theory of Numbers**, Hermann/Kershaw, 1973, ISBN 0901665061
Harvey Cohn, **Advanced Number Theory**, Dover publications, 1980, ISBN 048664023X
G.H. Hardy and E.M. Wright, **An Introduction to the Theory of Numbers**, OUP.
K.F. Ireland and M.I. Rosen, **A Classical Introduction to Modern Number Theory**, Springer
D. Marcus, **Number Fields**, Springer
Borevich and Shafarevich, **Number Theory**, Academic Press

* will serve as the main text excluding the last parts of the course. (Note that the previous editions are entitled just **Algebraic Number Theory**). This is a readable account with some historical details and motivation and some further material. Samuel is more abstract and concise but is otherwise also good. Cohn, also, covers much of the course nicely. The other books (in rough order of sophistication) are recommended for background and further reading.

Preliminary Reading Dip into any of the above books except perhaps the last three. Revise the ring theory from Algebra II.

Outline of course **Number Theory III/IV**

Aim: To provide an introduction to Algebraic Number Theory (Diophantine Equations and Ideal Theory).

<u>**Term 1**</u> (20 lectures)

A special case of Fermat: $x^3 + y^3 = z^3$.
Quick review of ring theory: Ideals (prime and maximal ideals). Factorization in monoids (prime *vs.* irreducible). Ring is Euclidean $\Rightarrow$ PID $\Rightarrow$ UFD. Preliminary discussion of ideal factorization. Examples of recovery of uniqueness of factorization using ideals. Applications, e.g. $x^2 + d = y^n$ with $d = 2$ and $n = 3$.
Fields: Algebraic number fields. Vector spaces over such. Field extensions: index; tower theorem; simple extensions and minimum polynomials; Conjugates over $\mathbb{Q}$. Homomorphisms of fields. Norms and traces over $\mathbb{Q}$. Matrix formulae for these.
Algebraic Integers: The several characterizations of algebraic integers O_K in a number field K; O_K is a ring. Integrality of norm and trace.
The Discriminant and Integral Bases: Definition, formal properties and calculation of the discriminant of a basis of a field over $\mathbb{Q}$. Determinant/index formula. Integral bases. Discriminants of lattices. Discriminant/index formula. Determination of an integral basis.
Factorization of Ideals: Fractional ideals. Unique factorization and inverses of ideals. The ideal group. Prime ideals. Factorization of (p) (Dedekind's theorem) and of other ideals. [Mention ramification and the discriminant.]
Quadratic Fields and Integers: Easy Euclidean quadratic fields [up to $\sqrt{29}$ if time]. Primes. Representation of integers by binary quadratic forms where there is unique factorization in the relevant quadratic field.

<u>**Term 2**</u> (18 lectures)

The Ideal Class Group: The class group. Minkowski's theorem. Finiteness and determination of the class group. Representation of integers by definite binary quadratic forms (problems rather than general theory). Further cases of $x^2 + d = y^n$.
Dirichlet's Unit Theorem: Proof of the theorem. Pell's equation revisited. Representation of integers by non-definite binary quadratic forms.
L-functions: Riemann zeta function, Dirichlet L-series, Zeta function for quadratic fields. Basic properties and analytic continuation.
Class number formula for quadratic fields

4H reading material: to be announced.

1.8.10 NUMERICAL DIFFERENTIAL EQUATIONS – MATH3081/MATH4221 (38 lectures)

Dr A. Felikson

Numerical Analysis has been defined as ‘the study of algorithms for problems of continuous mathematics’. An ordinary differential equation with appropriate initial or boundary conditions is an example of such a problem. We may know that the problem has a unique solution but it is rarely possible to express that solution as a finite combination of elementary functions. For example, how could we compute $y(3)$ to some specified accuracy, given that $y' + xy = 1$ and $y(0) = 0$? For this simple linear differential equation we can find an integrating factor and express $y(x)$ as an integral, but that only replaces one problem by another–try it. The vast number of methods available for approximating solutions of ordinary differential equations may be sub-divided into a small number of families, and within each family specific formulae may be derived systematically. Convergence and stability, in its various forms, are essential features of practical methods; techniques which allow us to investigate these properties will be established and used.

At the end of the first term and during the second term we will then be concerned with the numerical approximation of solutions of partial differential equations. Here we will see that the three main classes of second order partial differential equations—parabolic, elliptic, and hyperbolic—will each require a distinct approach. As in the first term, we will study convergence and stability, as well as practicality of various numerical schemes to approximate solutions.

Recommended Books

No particular book is recommended for purchase. The following are useful references:

K. W. Morton and D. F. Mayers, Numerical Solution of Partial Differential Equations: An Introduction, Second edition, Cambridge, 2005.
A. Iserles, A First Course in the Numerical Analysis of Differential Equations, Cambridge 1996.
E. Hairer, S. Norsett, and G. Wanner, Solving ordinary differential equations (vol 1), Springer 1993.
R.J. LeVeque, Finite Difference Methods for Ordinary and Partial Differential Equations, SIAM 2007.
J.C. Butcher, Numerical methods for ordinary differential equations, Wiley 2008.

Preliminary Reading: Read as much as you like of Chapter 3 of Plato, Concise Numerical Analysis, AMS 2003.

Reference for additional 4H Reading: To be announced in the lectures.

Aim: To build on the foundations laid in the level II Numerical Analysis module and to enable students to gain a deeper knowledge and understanding of two particular areas of numerical analysis.

Term 1 (20 lectures)

Numerical Solution of Ordinary Differential Equations: Introduction to numerical methods for initial-value problems. Local and global truncation errors, convergence. One-step methods, with emphasis on explicit Runge-Kutta methods. Practical algorithms for systems of equations. Linear multistep methods, in particular the Adams methods. Predictor- corrector methods and estimation of local truncation error. Stability concepts. BDF and stiff problems. Shooting methods for boundary-value problems.

Term 2 (18 lectures)

Approximation Theory: Piecewise polynomial approximation and spline functions. Best approximation, with emphasis on minimax and near-minimax polynomial approximations. Trigonometric polynomials, fast Fourier transformation and data compression. Higher-dimensional approximation and applications.

4H reading material: to be announced.

1.8.11 OPERATIONS RESEARCH III – MATH3141 (38 lectures)

Dr M. Troffaes / Prof M. Menshikov

As its name implies, operations research involves ”research on operations”, and it is applied to problems that concern how to conduct and coordinate the operations (activities) within an organization. The nature of the organization is essentially immaterial, and, in fact, OR has been applied extensively in such diverse areas as manufacturing, transportation, construction, telecommunications, financial planning, health care, the military, and public services to name just a few.

This course is an introduction to mathematical methods in operations research. Usually, a mathematical model of a practical situation of interest is developed, and analysis of the model is aimed at gaining more insight into the real world. Many problems that occur ask for optimisation of a function f under some constraints. If the function f and the constraints are all linear, the simplex method is a powerful tool for optimisation. This method is introduced and applied to several problems, e.g. within transportation.

Many situations of interest in OR involve processes with random aspects and we will introduce stochastic processes to model such situations. An interesting area of application, addressed in this course, is inventory theory, where both deterministic and stochastic models will be studied and applied. Further topics will be chosen from: Markov decision processes; integer programming; nonlinear programming; dynamic programming.

Recommended Books

* F.S. Hillier and G.J. Lieberman, **Introduction to Operations Research**, 8^{th} ed., McGraw-Hill 2004, ISBN 007123828x

This excellent book (softcover, £43.99) covers most of the course material (and much more), and the course is largely developed around this textbook. It is well written and has many useful examples and exercises. It also contains some PC software that can be used to gain additional insight in the course material. This is the best textbook we know of for this course (the 6^{th} and 7^{th} edition are equally useful).

There are plenty of books with ’operations research’, ’mathematical programming’, ’optimisation’ or ’stochastic processes’ in the title, and most of these contain at least some useful material. Be aware though that this course is intended to be an introduction, and most books tend to go rather quickly into much more detail.

Preliminary Reading

The first two chapters of the book by Hillier and Lieberman provide an interesting introduction into the wide area of operations research, and many of the other chapters start with interesting prototype examples that will give a good indication of the course material (restrict yourself to the chapters on topics mentioned above). Appendices 2 and 4 of this book could be read to refresh your understanding of matrices and the notion of convexity. Most other books within this area (see above) have introductory chapters that provide insight into the topic area, and the possible applications of OR.

Aim: To introduce some of the central mathematical models and methods of operations research.

Term 1 (20 lectures)

Introduction to Operations Research: Role of mathematical models, deterministic and stochastic OR.

Linear Programming: LP model; convexity and optimality of extreme points; simplex method; duality and sensitivity; special types of LP problems, e.g. transportation problem.

Networks: Analysis of networks, e.g. shortest-path problem, minimum spanning tree problem, maximum flow problem; applications to project planning and control.

Term 2 (18 lectures)

Introduction to Stochastic Processes: Stochastic process; discrete-time Markov chains.

Markov Decision Processes: Markovian decision models; the optimality equation; linear programming and optimal policies; policy-improvement algorithms; criterion of discounted costs; applications, e.g. inventory model.

Inventory Theory: Components of inventory models; deterministic models; stochastic models.

Further Topics Chosen From: Dynamic programming: Characteristics; deterministic and probabilistic dynamic programming.

Queueing theory: Models; waiting and service time distributions; steady-state systems; priority queues.

Integer programming: Model; alternative formulations; branch-and-bound technique; applications, e.g. knapsack problem, travelling salesman problem.

Nonlinear programming: Unconstrained optimisation; constrained optimisation (Karush-Kuhn-Tucker conditions); study of algorithms; approximations of nonlinear problems by linear problems; applications.

1.8.12 PARTIAL DIFFERENTIAL EQUATIONS III AND IV – MATH3291/MATH4041 (38 lectures)

The topic of partial differential equations (PDEs) is central to mathematics. It is of fundamental importance not only in classical areas of applied mathematics, such as fluid dynamics and elasticity, but also in financial forecasting and in modelling biological systems, chemical reactions, traffic flow and blood flow in the heart. PDEs are also important in pure mathematics and played a fundamental role in Perelman's proof of the Poincaré conjecture.

In this module we are concerned with the theoretical analysis of PDEs (the numerical analysis of PDEs is covered in the course Numerical Differential Equations III/IV). We will study first-order nonlinear PDEs and second-order linear PDEs, including the classical examples of Laplace's equation, the heat equation and the wave equation. For some simple equations with appropriate boundary conditions we will find explicit solutions. For equations where this is not possible we will study existence and properties of solutions. We will see for example that solutions of the heat equation have very different properties to solutions of the wave equation.

Recommended Books
No particular book is recommended for purchase. The following are useful references:

L.C. Evans (2010) *Partial Differential Equations*, 2nd edition, AMS (Chap. 2, 3);

Q. Han (2011) *A Basic Course in Partial Differential Equations*, AMS (Chap. 2, 4, 5, 6);
Y. Pinchover & J. Rubinstein (2005) *An Introduction to Partial Differential Equations*, Cambridge University Press (Chap. 2, 4, 7);

M. Shearer & R. Levy (2015) *Partial Differential Equations*, Princeton University Press (Chap. 3, 4, 5, 8, 13). **Preliminary Reading** To get a flavour of the course look at Chapters 3, 5 and 8 of Shearer & Levy.

Aim: To develop a basic understanding of the theory and methods of solution for partial differential equations.

Term 1 (20 lectures)

Introduction: examples of important PDEs, notation, the concept of well-posedness.

First-order PDEs and characteristics: the transport equation, general nonlinear first-order equations.

Conservation laws: models of traffic flow and gases, shocks and rarefactions, systems of conservation laws.

Second-order linear PDEs: examples and classification (elliptic, parabolic, hyperbolic).

Poisson's equation: fundamental solution.

Term 2 (18 lectures)

Laplace's equation: mean value formula, properties of Harmonic functions, maximum principle, Green's functions, energy method.

The heat equation: fundamental solution, maximum principle, energy method, infinite speed of propagation, properties of solutions.

The wave equation: solution formulas, energy method, finite speed of propagation, properties of solutions.

1.8.13 PROBABILITY III AND IV – MATH3211/MATH4131 (38 lectures)

Dr A. Wade / Dr O. Hryniv

Randomness is an essential feature of many models of real-world phenomena. Examples include locations of failures in a power grid, traffic fluctuations in the internet, flaws in ultra-pure materials, statistical behaviour of polymers to list just a few. While classical probability theory for sums of independent (or weakly dependent) variables can be used to describe some of these models, analysis and prediction of more dependent systems, especially for extreme events (like natural disasters, stock market crashes and bubbles, collective behaviour of social networks) require different approaches.

The purpose of this course is to attempt to bring your everyday intuition and common sense into agreement with the laws of probability and to explore how they can be used to analyse various applied models. We shall use elementary methods wherever possible, while discussing a variety of classical and modern applications of the subject ranging from random walks and renewal theory to records, extreme values and non-Gaussian limit results.

Recommended Books

G. R. Grimmett and D. R. Stirzaker, **Probability and random processes**, 3rd ed., OUP, 2001.
W. Feller, **An introduction to probability and its applications**, Volume I., Wiley, 1968.
S. M. Ross, **Introduction to probability models**, Academic Press, 1997.
A. Gut, **An Intermediate Course in Probability**, Springer, 2009.
A. N. Shiryaev, **Probability**, Springer, 1996.
R. Durrett, **Probability: theory and examples**, Duxbury Press, 1996.

Aim: This module continues on from the treatment of probability in 2H modules MATH2151 or MATH2161. It is designed to build a logical structure on probabilistic intuition; to study classical results such as the Strong Law of Large Numbers, the Central Limit Theorem, and some applications; to discuss some modern developments in the subject. Students completing this course should be equipped to read for themselves much of the vast literature on applications to biology, physics, chemistry, mathematical finance, engineering and many other fields.

Term 1 (20 lectures)

Introductory examples: from finite to infinite spaces.

Coin tossing and trajectories of random walks: ballot theorem, reflection principle, returns to the origin, excursions, arcsine laws.

Discrete renewal theorem in the lattice case and its applications.

Order statistics: order variables and their distributions. Exchangeability. Limiting behaviour of extrema. Some applications.

Stochastic order and its applications.

Non-Gaussian limits and their applications Some of: convergence to Poisson, exponential distributions; heavy-tailed distributions.

Term 2 (18 lectures)

General theory: probability spaces and random variables. σ-fields.

Integration: Expectation as Lebesgue integral. Inequalities. Monotone and dominated convergence. Characteristic functions.

Limit results: Laws of large numbers. Weak convergence. The central limit theorem. Kolmogorov 0-1 law. Large deviations.

Topics Chosen From: Random graphs, percolation, Gibbs distributions and phase transitions, randomized algorithms, probabilistic combinatorics, stochastic integral, information theory.

4H reading material: To be announced.

1.8.14 QUANTUM INFORMATION III – MATH3391 – (38 lectures)

Dr D. Smith / Prof S. Ross

Quantum mechanics differs in a fundamental way from classical mechanics. Quantum systems can be in a superposition of states, and measurements probabilistically select one state. This can be used in quantum computing where in some sense processing a superposition corresponds to parallel computing, but the catch is that we cannot measure all results. Systems can also be entangled even when they are separated, and this has interesting applications to communication, although it does not allow faster than light communication.

Quantum Information Theory describes how information can be stored and processed in quantum systems. It uses the properties of quantum systems such as superposition and mutually incompatible measurements in a fundamental way. As will be explained in this module, this gives the potential to allow completely secure communication between two parties (and this has been implemented), and a quantum computer is potentially for some tasks much more powerful than a classical computer (but there remain significant practical difficulties in constructing a useful device.) We will also explore how concepts such as entanglement can be quantified, and how it can be clearly determined that a quantum system is not simply a complicated classical system with randomness due to lack of knowledge.

Recommended Reading

There are many good textbooks and online resources. One standard comprehensive text is Nielsen and Chuang, *Quantum Computation and Quantum Information*. John Preskill has also provided an excellent set of lecture notes (covering much more than this module), see the link for Physics 219 from his webpage `http://www.theory.caltech.edu/~preskill/`

Aim: To provide an introduction to the application of quantum systems to processing information, specifically in terms of communication and computing. We will study the concept of quantum entanglement and demonstrate that quantum systems have properties that are fundamentally different from those of classical systems. We will see ways to quantify entanglement and how it can be used to achieve perfectly secure communication between two parties. We will also study some aspects of quantum information processing and investigate some examples of quantum algorithms.

Term 1 (20 lectures)

- Quantum Mechanics Introduction – Review of wave mechanics, introduction of Dirac notation and the density matrix. [4]
- Quantum Information – The qubit, Bloch sphere, bipartite systems and concept of pure and mixed states. [4]
- Quantum properties and applications – Superdense coding, teleportation, quantum key distribution, EPR paradox, Hidden variable theories and Bell inequalities. [6]
- Information, entropy and entanglement – Brief introduction to classical information theory including Shannon information and entanglement. Quantum entropy measures, von Neumann entropy, relative entropy and conditional entropy. [6]

Term 2(18 lectures)

- Classical computing – Turing machines (deterministic and probabilistic), universal gates/circuit models, very brief discussion of computational complexity. [4]
- Quantum computing – Quantum circuit model and universal gates, example algorithms (e.g. Grover's and Shor's), brief discussion of quantum computational complexity and comparison to classical examples (e.g. Shor's algorithm in context of RSA cryptography.) [8]
- Quantum error correction – Contrast to classical use of redundancy, examples of single qubit errors, use of entanglement to correct errors, example of Shor code. Discussion of error correction in quantum computing, including fault tolerant gates. [6]

1.8.15 QUANTUM MECHANICS III – MATH3111 (38 lectures)

Dr A. Donos / Dr M. Zamaklar

Quantum theory has been at the heart of the enormous advances that have been made in our understanding of the physical world over the last century, and it also underlies much of modern technology, e.g., lasers, transistors and superconductors. It gives a description of nature that is very different from that of classical mechanics, in particular being non-deterministic and 'non-local', and seeming to give a crucial role to the presence of 'observers'.

The course begins with a brief historical and conceptual introduction, explaining the reasons for the failure of classical ideas, and the corresponding changes needed to describe the physical world. It then introduces the defining postulates and reviews the necessary mathematical tools – basically linear vector spaces and second order differential equations. Applications to simple systems, which illustrate the power and characteristic predictions of the theory, are then discussed in detail.

Recommended Books

G. Auletta, M. Fortunato and G. Parisi, **Quantum Mechanics**, CUP 2009, ISBN 0521869633; £42.
C. Cohen-Tannoudji, B. Diu, F. Laloë, **Quantum Mechanics**, Hermann 1977, ISBN 2705658335; £39.
F. Mandl, **Quantum Mechanics**, Wiley 1992, ISBN 0471931551; £25.
L.I. Schiff, **Quantum Mechanics**, McGraw Hill 1969, ISBN 0070856435; £39.
A. Messiah, **Quantum Mechanics**, 2 Volumes, North Holland 2000, ISBN 0486409244; £20
P. Dirac, **The Principles of Quantum Mechanics**, OUP 4th Ed. 1981, ISBN 0198520115; £25
R. Shankar, **"Principles of quantum mechanics", New York : Plenum, c1980 ISBN 8181286863**
Landau and Lifshitz, **Quantum Mechanics. (Non-relativistic Theory)** Butterworth-Heinemann 1982, ISBN 0080291406; £35.

Any of the above covers most of the course. There are many other books which are quite similar.

Preliminary Reading

Because quantum theory is conceptually so different from classical mechanics it will be useful to have read descriptive, non-mathematical, accounts of the basic ideas, for example from the introductory chapters of the books above, or from general 'popular' accounts e.g., in Polkinghorne, 'The Quantum World'; Pagels, 'The Cosmic Code'; Squires, 'Mystery of the Quantum World', etc.

Aim: To give an understanding of the reasons why quantum theory is required, to explain its basic formalism and how this can be applied to simple situations, to show the power in quantum theory over a range of physical phenomena and to introduce students to some of the deep conceptual issues it raises.

Term 1 (20 lectures)

Problems with Classical Physics: Photo-electric effect, atomic spectra, wave-particle duality, uncertainty principle.

Formal Quantum Theory: Vectors, linear operators, hermitian operators, eigenvalues, complete sets, expectation values, commutation relations. Representation Theory, Dynamics: Schrödinger and Heisenberg pictures.

Spectra of Operators: Position operator, Harmonic Oscillator, Angular momentum.

Term 2 (18 lectures)

Waves and the Schrödinger Equation: Time-dependent and time-independent Schrödinger equation. Probability meaning of $|\Psi|^2$. Currents. Plane-waves, spreading of a wave packet.

Applications in One Dimension: Square well, harmonic oscillator, square barrier, tunnelling phenomena.

Three dimensional problems Three-dimensional harmonic oscillator, Hydrogen atom, other systems.

1.8.16 STATISTICAL MECHANICS III/IV – MATH3351/MATH4231 (38 lectures)

TBA / Dr B. Chakrabarti

Statistical Mechanics seeks to use statistical techniques to develop mathematical methods necessary to deal with systems with large number of constituent entities, like a box of gas with which comprises of a large number of individual gas molecules. It provides a framework for relating the microscopic properties of individual atoms and molecules to the macroscopic or bulk properties of materials that can be observed in everyday life, therefore explaining thermodynamics as a natural result of statistics and mechanics (classical and quantum) at the microscopic level.

We will start by discussing basic concepts in thermodynamics and then proceed to see how these ideas can be derived from a statistical viewpoint. As we go along, we will learn to apply the concepts to various interesting physical phenomena, such as neutron stars, Bose-Einstein condensation, phase transitions, *etc*...

Recommended Books

R. K. Pathria, **Statistical Mechanics**, Second Edition, Elsevier 1996.
K. Huang, **Statistical Mechanics**, Wiley; 2 edition 1987.
F. Reif, **Fundamentals of Statistical and Thermal Physics**, McGraw-Hill, 1975.
R. Blowey and M. Sanchez, **Introductory Statistical Mechanics**, Oxford science publications, 1999.
L. D. Landau and E. M. Liftshitz, **Statistical Physics**, 3rd Edition, Part 1, Buttterworth Heinemann, 1980.

* This will be the main textbook for the course. Pathria covers most of the material as well. Reif is a good source for basics of thermodynamics. Landau and Liftshitz is a classic book and an excellent resource for advanced topics.

Preliminary Reading

Dip into any of the above books except perhaps the last one. In particular, the first few chapters of Reif should provide a good starting point.

Aim: To develop a basic understanding of dynamics and behaviour of systems with a large number of constituents.

<u>**Term 1**</u> (20 lectures)

Thermodynamics: Thermal equlibrium, the laws of thermodynamics. Equations of state, ideal gas law.

Classical statistical mechanics: Statistical basis of thermodynamics: microstates, macrostates and the thermodynamics limit. Ideal gas. Gibbs paradox and entropy. Microcanonical, canonical and grand-canonical ensembles.

<u>**Term 2**</u> (18 lectures)

Distributions and identical particles: Maxwell-Boltzmann distribution. Bose and Fermi distributions, parastatistics.

Physical phenomena: Ideal Bose and Fermi gases. Black-body radiation, magnetisation, neutron stars (Chandrashekar limit), superfluidity, Bose-Einstein condensation.

Phase transitions: Mean field theory, Landau-Ginzburg theory of phase transitions.

1.8.17 STATISTICAL METHODS III – MATH3051 (38 lectures + 8 R practicals)

Dr J. Einbeck

The course introduces widely used statistical methods. The course should be of particular interest to those who intend to follow a career in statistics or who might choose to do a fourth year project in statistics. Having a particular emphasis on the intersection of theory and practice, the learning objective of the course includes the ability of performing hands-on data analysis using the statistical programming language R. Therefore, four computer practicals will be held in each of Michaelmas and Epiphany term. Towards the end of each term, a practical examination component will be held, each of which contributes 15% towards the total examination mark.

Topics include: statistical computing using `R`; multivariate analysis (in particular, principal component analysis); regression (linear model: inference, prediction, variable selection, influence, diagnostics, outliers,); analysis of designed experiments (analysis of variance); extensions to transformed, weighted, and/or nonparametric regression models;

There is no one recommended book but the books below more than cover the course material; in particular those by Weisberg and Krzanowski provide (in conjunction) a good coverage in an accessible style. The book by Kutner et al. is quite voluminous but worth of consideration for those who prefer a detailed step-by-step description of the methods.

Recommended Books

M.J. Crawley, **The R book**, Wiley 2007, ISBN 0470510242.
W.J. Krzanowski, **Principles of Multivariate Analysis: A User's Perspective**, OUP Oxford 2000, ISBN 0198507089.
M. Kutner, C. Nachtsheim, J. Neter, W. Wasserman and W. Li, **Applied Linear Statistical Methods** (several editions with different combinations of authors 1964–2005), ISBN 007310874X.
K.V. Mardia, J.T. Kent, and J.M. Bibby, **Multivariate Analysis**, Academic Press 1979, ISBN 0-12-471250-9.
T. Raykov, **Basic Statistics - an introduction with R**, Rowman & Littlefield Publishers 2012, Access via Durham MyiLibrary, ISBN 9781283833837.
J.A. Rice, **Mathematical Statistics and Data Analysis**, Brooks/Cole 2006, 3rd ed. ISBN 0495110892 (a used, i.e. cheap, second edition is also appropriate).
S. Weisberg, **Applied Linear Regression**, Wiley 2005, ISBN 0471879576.

Preliminary Reading

Chapters 3.1-3.4, 8.1-8.4, 12, and 14 in Rice's book **Mathematical Statistics and Data Analysis**, Brooks/Cole 2006.

Aim: To provide a working knowledge of the theory, computation and practice of statistical methods, with focus on the linear model.

Term 1 (19 lectures + 3 R practicals)

Basics: Statistical computing in R, matrix algebra, multivariate probability and likelihood, multivariate normal distribution.

The linear model: Assumptions, estimation, inference, prediction, analysis of variance, designed experiments, model selection.

Term 2 (17 lectures + 3 R practicals)

Regression diagnostics: influence, outliers, lack-of-fit.

Introduction to multivariate analysis: Variance matrix estimation, Mahalanobis distance, principal component analysis; dimension reduction.

Extensions: Basics of transformed, weighted, and/or nonparametric regression models.

1.8.18 TOPICS IN STATISTICS III and IV – MATH3361/MATH4071 (38 lectures)

The module presents a number of widely used statistical methods, some building on material in Statistical Concepts II and others on material in Statistical Methods III which is a pre-/co-requisite.

The module should be of particular interest to those who intend to follow a career in statistics or who might choose to do a fourth year project in statistics. It is intended to broaden the range of contexts in which you will be able to function as a statistician and to build upon the linear model intuition developed in Statistical Methods III.

Key topics are: the generally applicable method of maximum likelihood estimation in the context of models with multiple parameters; contingency table analysis which nicely complements the material on modelling continuous variables in Statistical Methods III; generalized linear models which extend the linear model ideas in SMIII to a wide class of data scenarios. This includes regression problems with categorical response, as used for instance in the banking sector for credit scoring. Time permitting, an advanced topic will be studied towards the end of the course.

Recommended Books

Since the module is a selection of topics in statistics, no one book covers the material. The following are recommended for reference and selected material from them will be made available via DUO in due course.

Y. Pawitan, **In All Likelihood: Statistical Modelling and Inference Using Likelihood**, Oxford, 2001, ISBN 0198507658

A. Agresti, **An Introduction to Categorical Data Analysis** (2nd ed), Wiley, 2007, ISBN 0471226181

A.J. Dobson, A. Barnett and K. Grove, **An Introduction to Generalized Linear Models** (3rd ed), CRC Press, 2008, ISBN 1584889500

L. Fahrmeir & G. Tutz, **Multivariate Statistical Modelling Based on Generalized Linear Models** (2nd ed), Springer, 2001, ISBN 0387951873

Level IV reading material reference: to be specified by the lecturer in due course.

Aim: To provide a working knowledge of the theory, computation and practice of a variety of widely used statistical methods.

Term 1 (20 lectures)

Likelihood estimation: Likelihood and score functions for multi-parameter models, Fisher information, confidence regions, method of support, likelihood ratio tests, profile likelihood, Akaike and Bayes Information Criteria (AIC & BIC).

Contingency tables: Sampling models, log-linear modelling, iterative proportional fitting, model selection, goodness of fit.

Term 2 (18 lectures)

Generalised linear models: Framework, exponential families, likelihood and deviance, standard errors and confidence intervals, prediction, analysis of deviance, residuals, over-dispersion.

Advanced topic: One of multivariate analysis, time series analysis, or medical statistics.

1.8.19 TOPOLOGY III – MATH3281 – (38 lectures)

Dr D. Schuetz / Prof J. Hunton

Topology is a mathematical theory which explains many interesting natural phenomena: it helps to study equilibrium prices in economics, instabilities in behaviour of dynamical systems (for example, robots), chemical properties of molecules in modern molecular biology and many other important problems of science. Topological methods are used in most branches of pure and applied mathematics.

The course is an introduction to topology. We shall start by introducing the idea of an abstract topological space and studying elementary properties such as continuity, compactness and connectedness. A good deal of the course will be geometrical in flavour. We shall study closed surfaces and polyhedra, spend some time discussing winding numbers of planar curves and their applications. Orbit spaces of group actions will provide many interesting examples of topological spaces.

Elements of algebraic topology will also appear in the course. We shall study the fundamental group and the Euler characteristic and apply these invariants in order to distinguish between various spaces. We shall also discuss the concept of homotopy type; this material will be further developed in the course 'Algebraic Topology IV'.

Recommended Books

M.A. Armstrong, **Basic Topology**, Springer-Verlag 1983, ISBN 3540908390
R. H. Crowell and R. H. Fox, **Introduction to Knot Theory**, 1963.
W. A. Sutherland, **introduction to metric and topological spaces**, 2009.

Preliminary reading

To get an idea of what topology is about one may consult the first two chapters of Armstrong. Any of the following two books may also serve as an enjoyable introduction to topology:

W. G. Chinn and N.E. Steenrod, **First Concepts of Topology**, Random House, New York, 1966
C.T.C. Wall, **A Geometric Introduction to Topology**, Addison Wesley, Reading, Mass., 1972.

Aim: To provide a balanced introduction to Point Set, Geometric and Algebraic Topology, with particular emphasis on surfaces.

Term 1 (20 lectures)

Topological Spaces and Continuous Functions: Topological spaces, limit points, continuous maps, homeomorphisms, compactness, product topology, connectedness, path-connected spaces, quotient topology, graphs and surfaces.

Topological Groups and Group Actions: Orthogonal groups, connected components of $O(n)$, the concept of orientation, topological groups, quaternions, group actions, orbit spaces, projective spaces, lens spaces.

Term 2 (18 lectures)

Elements of geometric topology:

Classification of graphs up to homeomorphism. Connected sums of surfaces. Orientable and non-orientable surfaces. The topological classification of compact surfaces. Polyhedra, triangulations of topological spaces, the topological invariance of the Euler characteristic, properties of the Euler characteristic.

Basic concepts of homotopy theory: The winding number of loops and applications. Homotopic maps and homotopy equivalence of spaces. Classification of graphs up to homotopy.

1.8.20 ADVANCED QUANTUM THEORY IV – MATH4061 (38 lectures)

Dr M. Zamaklar / Dr K. Peeters

The course will introduce Quantum Field Theory (QFT) by bringing together concepts from classical Lagrangian and Hamiltonian mechanics, quantum mechanics and special relativity. It also provides an elementary introduction to string theory, both as a simple two-dimensional QFT and as a way to go beyond QFT concepts.

The course will begin with a reminder of classical field theory concepts. We then go on to a discussion of free and interacting quantum field theories in the operator formalism, explain Feynman diagrams and the computation of scattering amplitudes. Throughout, simple examples will be used to emphasise the conceptual ingredients rather than the computational technicalities.

In the second term, we begin with an explanation of the path integral formulation of quantum theory, both in relativistic quantum mechanics and in quantum field theory. String theory follows next, and we will discuss its spectrum, symmetries and dualities. We will see how self-consistency conditions lead to a preferred number of spacetime dimensions and the existence of gauge fields and gravitons (providing a connection to general relativity). The last part of the course is about scale dependence, renormalisation and the renormalisation group.

The course is complementary to the PHYS4181 Particle Theory module in the sense that it focuses more on conceptual and mathematical foundations. It should prove interesting and useful especially to students who want to continue to pursue an interest in what is currently understood of the fundamental nature of matter.

Recommended Books

Typed lecture notes will be provided. In addition, there are many QFT books and string theory texts so you will find no shortage of material in the library. Some suggestions:

Anthony Zee, **Quantum Field Theory in a Nutshell**, Princeton 2010, ISBN 0691140340.

Brian Hatfield, **Quantum Field Theory of Point Particles and Strings**, Perseus 1999, ISBN 0201360799.

Mark Srednicki, **Quantum Field Theory**, Cambridge 2007, ISBN 0521864496.

Barton Zwiebach, **A First Course in String Theory**, Cambridge 2004, ISBN 0521831431.

Michael Green, John Schwarz & Edward Witten, **Superstring theory, Volume 1, Introduction**, Cambridge 1988, ISBN 0521357527.

Aim: To introduce quantum field theory using the operator formalism as well as path integrals, and to apply it to string theory, developing it sufficiently to show that its spectrum includes all elementary particles thus unifying the fundamental forces.

Term 1 (20 lectures)

Action principles and classical theory: Review of Lagrangian formulation of classical field theory. Symmetries, equations of motion, Lagrangian and Hamiltonian methods, Noether's theorem.

Quantisation of free scalar fields: Multi-particle quantum mechanics, canonical quantisation of free scalar fields, Fock space, anti-particles, propagators, causality.

Interacting quantum fields: Evolution operators, perturbative expansion, Wick's theorem, Feynman diagrams in position and momentum space, LSZ reduction, scattering matrix, cross sections.

Term 2 (18 lectures)

Path integrals: Relativistic particle in the world-line formulation, generating functionals, diagrammatic expansion, zero-dimensional quantum field theory.

String theory: World-line action for free relativistic particle, formulation with intrinsic metric. Nambu-Goto action and Polyakov action and equations of motion, symmetries, boundary conditions, simple classical solutions, quantisation, Virasoro algebra, physical states, connection to general relativity, T-duality.

Renormalisation and scale dependence: Regularisation methods, renormalisation, power counting, renormalisation group flow.

1.8.21 ALGEBRAIC TOPOLOGY IV – MATH4161 (38 lectures)

Dr A. Lobb / Dr D. Schuetz

The basic method of Algebraic Topology is to assign an algebraic system (say, a group or a ring) to each topological space in such a way that homeomorphic spaces have isomorphic systems. Geometrical problems about spaces can then be solved by 'pushing' them into algebra and doing computations there. This idea can be illustrated by the theory of fundamental group which is familiar from the Topology III course.

In the course 'Algebraic Topology' we meet other, more sophisticated theories such as singular homology and cohomology. The course begins with an introduction to cell complexes, a convenient class of topological spaces which contain many important examples and which are amenable to homology calculations. Properties of singular homology are studied and used to prove important theorems such as the Brouwer Fixed Point Theorem and the Jordan Curve Theorem.

In the second term cohomology is introduced, which appears formally very similar to homology, but adds useful structure to the theory. Applied to manifolds, the full power of these theories lies in their interplay, which culminates in the duality theorems of Poincaré and Alexander. As an application we show that non-orientable surfaces such as the real projective plane and the Klein bottle cannot be embedded in 3-dimensional Euclidean space.

Recommended Books

Allen Hatcher, **Algebraic topology**, Cambridge university press, 2002; This book is available free of charge from `www.math.cornell.edu/~hatcher`

M.A. Armstrong, **Basic topology**, Springer-Verlag, 1983

G. Bredon, **Topology and Geometry**, Springer-Verlag, 1993

A. Dold, **Lectures on Algebraic topology**, Springer-Verlag, 1980

W. Fulton, **Algebraic topology (a first course)**, Springer - Verlag, 1995.

E. Spanier, **Algebraic topology**, 1966

Aim: The module will provide a deeper knowledge in the field of topology (a balanced introduction having been provided in Topology III (MATH3281))

<u>**Term 1**</u> (20 lectures)

Homotopy properties of cell complexes: Cell complexes, main constructions (mapping cones and cylinders, products), examples.

Elements of homological algebra: Chain complexes, homology, chain homotopy, exact sequences, Euler characteristic.

Homology theory of topological spaces: Singular homology of topological spaces, homotopy invariance, Mayer - Vietoris sequences, relation between homology and the fundamental group, geometric interpretation of homology classes, homology groups of cell complexes.

Applications: Brouwer Fixed Point Theorem, Jordan Curve Theorem, Invariance of Domain.

<u>**Term 2**</u> (18 lectures)

Cohomology theory of topological spaces: Singular cohomology of topological spaces, Statement of the Universal Coefficient Theorem, cup products and ring structure, Künneth formula.

(Co)homology of manifolds: Fundamental classes, orientations in terms of homology, intersection numbers, Poincaré duality.

Applications: Alexander duality, Non-embeddability of non-orientable closed surface in Euclidean 3-space.

1.8.22 ELLIPTIC FUNCTIONS IV – MATH4151 (38 lectures)

Dr J. Funke / Dr J. Parker

Modular forms and elliptic functions are closely related topics in complex analysis and are ingredients of much current research, from number theory and geometry to representation theory and theoretical physics. In particular, modular forms play an increasingly central role in modern number theory, for example in the celebrated proof of Fermat's last theorem. The name elliptic functions arises from the problem of finding the arc length of an ellipse, which leads to an integral that cannot be evaluated by elementary functions. The idea of 'inverting' such integrals, rather like finding $\sin^{-1} x$ as an integral, led to the theory of doubly periodic functions. The development of that theory, by Jacobi and Weierstrass and others, was one of the high points of nineteenth century mathematical achievement.

In this course we will cover the classical theory of elliptic functions and modular forms and touch on a few applications. We will make extensive use of techniques from complex analysis covered in MATH2011, which is a prerequisite for this course. In addition we will use a bit of geometry and number theory.

Recommended Books

J. Serre, **A Course in Arithmetic**, Springer 1996.
N. Koblitz, **Introduction to Elliptic Curves and Modular Forms**, Springer 1993.
F. Diamond & J. Shurman, **A First Course in Modular Forms**, Springer 2007.
G.A. Jones & D. Singerman, **Complex Functions**, Cambridge University Press 1987.
J.V. Armitage & W.F. Eberlein, **Elliptic Functions**, Cambridge University Press 2006.
Derek F. Lawden, **Elliptic Functions and Applications**, Springer-Verlag, New York 1989.
E. Freitag & R. Busam, **Complex Analysis**, Springer 2009.

All of these books contain more material than is needed for the course. For the first term, the book by Serre is a true classic. We will be concerned with Chapter VII, where modular forms are discussed. On occasion we will also consult the books by Koblitz and Diamond-Shurman which are slightly more advanced.

For the second term, we will use the (well-written and inexpensive) book by Jones and Singerman, Chapters 1, 3 and 6. The book by Armitage and Eberlein takes a different approach to elliptic functions. It also is a good source of applications.

The book by Freitag and Busam covers a lot of material in complex analysis. Elliptic functions are covered in Chapter V; modular forms in Chapters VI and VII.

Preliminary Reading

For Modular Forms: Revise the concepts of uniform convergence and Weierstrass M-test; read Section 1 of Chapter VII in Serre;

For Elliptic Functions: Read carefully Chapter 1 of Jones & Singerman and try problems 1E to 1N at the end. Then browse through Chapter 3.

Outline of course **Elliptic Functions IV**

Aim: To introduce the theory of modular forms of one complex variable and multiply-periodic functions and to develop and apply it.

<u>**Term 1: Modular Forms**</u> (20 lectures)

Basics of the Riemann ζ function: Definition, Euler product, values at even integers

The modular group: The modular group $SL_2(\mathbb{Z})$. Fundamental domain. Relationship to lattices in the complex plane.

Modular forms for the full modular group: Definition of M_k and S_k. Eisenstein series of weight $k \geq 4$, relationship to elliptic functions; Fourier expansion. Eisenstein series of weight 2, $\eta(\tau)$, $\Delta(\tau)$, Ramanujan τ-function.

The $k/12$-formula: Proof. Dimension formula for M_k, S_k. Structure theorem for $\oplus_{k\geq 0} M_k$. The j-invariant and its properties.

Modular forms for congruence subgroups

Theta series: Jacobi theta series ϑ. Theta inversion. Application to sums of squares. Theta series for unimodular lattices.

L-functions: Definition. Analytic continuation and functional equation for modular L-functions; in particular for Riemann zeta function.

Hecke operators

<u>**Term 2: Elliptic Functions**</u> (18 lectures)

Motivation: Elliptic integrals, simple pendulum.

General Properties of Elliptic Functions: Review of necessary complex analysis. Periodic functions. Necessary conditions on zeros, poles and residues. Topological properties of elliptic functions.

Weierstrass Functions: Convergence / divergence of $\sum' |\omega|^\alpha$. Weierstrass functions $\sigma(z)$, $\zeta(z)$, $\wp(z)$. Differential equation satisfied by $\wp(z)$. Expression of general elliptic functions in terms of $\wp(z)$ and $\wp'(z)$. Construction of elliptic functions with given zeros and poles, and with given principal parts. The addition theorem for $\wp(z)$.

1.8.23 GENERAL RELATIVITY IV – MATH4051 (38 lectures)

TBA / Prof R. Ward

A century ago, Einstein invented a geometric model of gravity, namely General Relativity. In addition to being mathematically beautiful, it provides an extraordinarily accurate description of gravitational phenomena. The key idea is that gravity is a manifestation of the geometry of space-time.

The first half of this course concentrates mainly on setting up the necessary geometrical structure, and introducing Einstein's equations which describe how matter curves space-time. In the second half, we shall see how all this is applied, for example to various solutions of Einstein's equations, and their interpretation; classical solar-system tests of GR, such as the bending of light around the sun and the perihelion shift of Mercury; black holes, event horizons and Penrose diagrams; and cosmology which studies the universe on a large scale.

Recommended Books.

L.P. Hughston and K.P. Tod, **An Introduction to General Relativity**, Cambridge 1990.
S. Carroll, **Lecture Notes on General Relativity**, Pearson 2014.
B.F. Schutz, **A First Course in General Relativity**, Cambridge 2009.
R. d'Inverno, **Introducing Einstein's Relativity**, Oxford 1995.
R.M. Wald, **Space, Time and Gravity**, Chicago 1992.
M.P. Hobson, G.P. Efstathiou & A.N. Lasenby, **General Relativity**, Cambridge 2006.
I.R. Kenyon, **General Relativity**, Oxford 1990.
T.P. Cheng, **Relativity, Gravitation and Cosmology**.
J. Hartle, **Gravity: an introduction to Einstein's General Relativity**, Addison-Wesley 2003.
G.F.R. Ellis & R.M. Williams, **Flat and Curved Space-Times**.
L. Ryder, **Introduction to General Relativity**.
R.J.A. Lambourne, **Relativity, Gravitation and Cosmology**.
W. Rindler, **Relativity : Special, General and Cosmological**.
There are also several online texts available.

Preliminary Reading.

The Wikipedia articles on "Introduction to general relativity" and "General relativity" give an overview, and many links. The first few chapters of the books above, for example Hughston & Tod and Schutz, would be useful.

Aim: To appreciate General Relativity, one of the fundamental physical theories. To develop and exercise mathematical methods.

Term 1 (20 lectures)

Introduction to General Relativity: differences between GR & SR, gravity as geometry, equivalence principle.

Special Relativity: spacetime diagrams, line element, vectors and tensors, electromagnetism, stress-energy tensor.

Differential Manifolds: spacetime is a manifold, coordinates and coordinate transformations, tangent vectors, tensors revisited.

Metric: distance relationships, light cones, Riemann normal coordinates.

Covariant Derivative: inadequacy of partial derivatives, parallel transport, connection coefficients, differentiating tensors, metric connection, geodesics.

Curvature: Riemann tensor, characterisation of flat space, parallel transport around closed curves, commutation formulae, Bianchi identity, Einstein tensor, geodesic deviation.

Term 2 (18 lectures)

General Relativity: equivalence principle, physics in curved spacetime, Einstein's equations, linearized theory and Newtonian limit, Einstein-Hilbert action.

Black Holes: spherical symmetry, Schwarzschild solution, geodesics, solar-system applications, event horizon and Kruskal coordinates, black hole formation.

Cosmology: isotropy and homogeneity, FRW metric, examples of cosmologies, Hubble law, particle horizons.

1.8.24 RIEMANNIAN GEOMETRY IV – MATH4171 (38 lectures)

Dr W. Klingenberg

This course is based on the course Differential Geometry III as a prerequisite.

In the first term we introduce Riemann's concept of a manifold as a space with locally Euclidean coordinates and an intrinsic method of measuring distances and angles and discuss it in several examples. We also introduce geodesics via an variational approach and discuss parallel transport of vector fields along curves.

The second term is concerned with the notion of curvature and its influence on the geometry of the underlying space. We study spaces of constant curvature, present a global curvature comparison theorem (Bonnet-Myers) and discuss applications.

The aim of this course is to develop acquaintance with more general geometric spaces than just the Euclidean space and to the geometric meaning of curvature.

Possible Books:

M. P. Do Carmo: **Riemannian Geometry**, Birkhäuser, ISBN 0-8176-3490-8

F. Morgan: **Riemannian Geometry: A Beginner's Guide**, Jones and Bartlett Publishres, ISBN 0-86720-242-2

S. Gallot, D. Hulin, J. Lafontaine: **Riemannian Geometry**, Springer, ISBN 0-387-52401-0

J. Lee: **Riemannian Manifolds, An Introduction to Curvature**, Springer, ISBN 0-387-98271-X

J. Cheeger, D. G. Ebin: **Comparison Theorems in Riemannian Geometry**, Elsevier Science, ISBN 0-444-107649

Aim: The aim of this course is to develop acquaintance with more general geometric spaces than just the Euclidean space and to the geometric meaning of curvature.

Term 1 (20 lectures)

- From submanifolds to **abstract manifolds** via examples: surfaces of revolution, projective space, Grassman manifolds, hyperbolic space, matrix groups
- Tangent vectors and **tangent space**, computations in examples (e.g. in matrix groups), vectorfields and computations of Lie brackets
- examples of **Riemmanian manifolds**
- length of curves, Riemannian manifolds as metric spaces
- local and global properties of **geodesics**, first variation formula, **Levi-Civita connection**, parallel transport, discussion of examples (e.g. hyperbolic space, matrix groups)
- geodesics on surfaces of revolution, Clairaut's theorem

Term 2 (18 lectures)

- different curvature notions and computations thereof: **Riemmanian curvature tensor**, sectional curvature, Ricci curvature, scalar curvature
- spaces of constant curvature, the Cartan-Ambrose-Hicks theorem
- integration on Riemannian manifolds, volume calculations in spaces of constant curvature
- the second variation formula and **Bonnet-Myers theorem** as a global comparison result
- applications (e.g., the n-torus does not admit a metric of positive curvature; the global 2-dimensional Gauß-Bonnet theorem from differential geometry implies same statement for 2-torus)