Jacobs University
School of Engineering and Science
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Spring Term 2009

Problem Set 10

Analysis II

Homework Problems

10.1. (a) Show that the set of points of $[0,1]$ containing no digit 1 in its decimal expansion has measure zero.

(b) Find the Lebesgue measure of the set of point of $[0,1]$ containing all the digits $1,2,\dots,9$.

10.2. Show that any closed subset of $\mathbb{R}$ of measure zero is nowhere dense.

10.3. Let $f\colon A\to\mathbb{R}$ be measurable, and $g\colon f(A)\to\mathbb{R}$ be continuous. Show that $g\circ f$ is measurable.

10.4. Let $f:\mathbb{R}^n\to\mathbb{R}$ satisfy the following condition: $\{x\mid f(x)=c\}$ is measurable for any c. Does this imply that f is measurable?

10.5. Let C be the Cantor set. Define the *Cantor function* $\varphi:[0,1]\to[0,1]$ as follows:

If $x=\sum\limits_{n=1}^{\infty}\frac{a_n}{3^n}$ with $a_n=0$ or 2, then

$$\varphi(x)=\varphi\left(\sum_{n=1}^{\infty}\frac{a_n}{3^n}\right)=\sum_{n=1}^{\infty}\frac{a_n}{2}\frac{1}{2^n}$$

that is, if a_n is the nth ternary digit for x, then nth binary digit for $\varphi(x)$ is $a_n/2$. Extend $\varphi(x)$ to $[0,1]$ by setting

$$\varphi(x)=\sup\{\varphi(y)\mid y\in C, y<x\}$$

(a) Show that $\varphi(C)=[0,1]$ (in particular, the image of zero set has measure 1).

(b) Show that $\varphi(x)$ is increasing and continuous on $[0,1]$, and $\varphi'(x)=0$ almost everywhere on $[0,1]$.

10.6. ($\star$) Show that every subset of $\mathbb{R}$ of positive measure contains a non-measurable subset.

Due Date: Friday, May 15, at the beginning of the class.