Jacobs University
School of Engineering and Science
Pavel Tumarkin, Yauhen (Zhenya) Mikulich

Spring Term 2009

Problem Set 2

Analysis II

Homework Problems

2.1. Consider a motonic function $f : M \to \mathbb{R}$.

(a) Show that f has both one-sided limits at every point of M.

(b) Show that the number of discontonuity points of f is countable.

In the sequel $\alpha : [a,b] \to \mathbb{R}$ is an increasing function, $f : [a,b] \to \mathbb{R}$ is bounded.

For every partition P of $[a,b]$ choose points $t_1, \dots, t_n$ such that $x_{i-1} \le t_i \le x_i$, and define

$$S(P, f, \alpha) = \sum_{i=1}^{n} f(t_i)\Delta\alpha_i.$$

Define also $\mu(P) = \max\limits_{i \le n} \Delta x_i$. We say that

$$\lim_{\mu(P)\to 0} S(P, f, \alpha) = A$$

if for any $\varepsilon > 0$ there is $\delta > 0$ such that $\mu(P) < \delta$ implies $|S(P, f, \alpha) - A| < \varepsilon$ for *all* admissible choices of t_i. In case of $\alpha(x) = x$ we denote $S(P, f, \alpha) = S(P, f) = \sum f(t_i)\Delta x_i$.

2.2. (a) *Equivalent definition of Riemann integral*

Show that the limit $\lim\limits_{\mu(P)\to 0} S(P, f) = A$ exists if and only if $f \in \mathcal{R}$; prove that in this case $\int\limits_a^b f(x)\,dx = A$;

(b) show that if the limit $\lim\limits_{\mu(P)\to 0} S(P, f, \alpha) = A$ exists then $f \in \mathcal{R}(\alpha)$, and $\int\limits_a^b f\,d\alpha = A$;

(c) show that if f is continuous on $[a,b]$, then $\lim\limits_{\mu(P)\to 0} S(P, f, \alpha) = \int\limits_a^b f\,d\alpha$.

($\star$) show that if α is continuous on $[a,b]$, $f \in \mathcal{R}(\alpha)$ if and only if the limit $\lim\limits_{\mu(P)\to 0} S(P, f, \alpha)$ exists.

2.3. *Mean Value Theorem*

Let f be continuous on $[a,b]$. Then there exists $c \in [a,b]$ such that $\int\limits_a^b f\,d\alpha = f(c)(\alpha(b) - \alpha(a))$.

2.4. Construct a function $f \in \mathcal{R}$ on $[a,b]$ such that f has no antiderivative.

2.5. ($\star$) Construct a function $f \notin \mathcal{R}$ on $[a,b]$ such that f has antiderivative.

Due Date: Wednesday, February 25, at the beginning of class.