Jacobs University
School of Engineering and Science
Pavel Tumarkin, Yauhen (Zhenya) Mikulich

Spring Term 2009

Problem Set 4

Analysis II

Homework Problems

Throughout this assignment, all the sets are subsets of metric space.

4.1. *Open and closed sets*

(a) Let $M \subset \mathbb{R}$ be an open set. Show that either $M = \mathbb{R}$ or M is a union of a countable (or finite) set of mutually disjoint open intervals and rays.

(b) Prove that closure $\overline{M} = M \cup M'$ of M is closed, and every closed set containing M contains $\overline{M}$.

($\star$) Let X be a complete metric space, and (B_n) be a sequence of closed balls, $B_n \subset B_{n-1}$. Is it true that $\bigcap\limits_{n=1}^{\infty} B_n \neq \emptyset$?

4.2. *Perfect sets*

A subset $M \subset M_1$ is called *dense in* M_1 if $M_1 \subset M'$. In particular, M is called *dense-it-itself* if any point of M is a limit point. (A point of M which is not a limit point is called *isolated point*. So, a set is dense-in-itself iff it contains no isolated points). A set M is called *perfect* if $M = M'$.

(a) Show that M is perfect if and only if M is closed and dense-in-itself.

($\star$) Show that any perfect $M \subset \mathbb{R}$ is uncountable.

4.3. *Contractions*

Let X be a metric space. A map $f : X \to X$ is called a *contraction* if there exists a positive $c < 1$ such that $|f(x) - f(y)| \leq c|x - y|$ for any $x, y \in X$.

(a) Show that any contraction is continuous.

(b) Show that if X is complete, then any contraction has a unique fixed point in X.

(c) Will the statement of (b) remain true if $f(x)$ is not a contraction, but satisfies assumption $|f(x) - f(y)| < |x - y|$ for all $x \neq y$?

4.4. *Compact sets*

Let X, Y be metric spaces, $f : X \to Y$ is continuous and $K \subset X$ is compact.

(a) Show that $f(K)$ is compact.

(b) Show that if restriction of f on K is injective then it is a homeomorphism.

4.5. *Cantor set*

Consider a sequence of sets

$$I_1 = [0, 1], \qquad I_{n+1} = \{x \in \mathbb{R} \,|\, 3x \in I_n \text{ or } 3x - 2 \in I_n\}, \ n \geq 1.$$

The set $C = \bigcap_{n \in \mathbb{N}} I_n$ is called *Cantor set*.

(a) Show that $I_{n+1} \subset I_n$ for any n.

(b) Give a description of C in terms of 3-base numeral system.

(c) Show that C is not countable.

(d) Which of the numbers $0, \frac{1}{3}, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{3}{4}$ belong to C?

(e) Is C open? closed? dense-in-itself? perfect?

4.6. *Nowhere dense sets*

A point x of $E \subset X$ is called *interior point of* E if E contains some open neighborhood of x. A set E is called *nowhere dense* if $\overline{E}$ has no interior points.

(a) Show that any subset of nowhere dense set is nowhere dense.

(b) Show that a union of finite number of nowhere dense sets is nowhere dense.

(c) Which of the following sets are nowhere dense: $\emptyset$; finite set; $\mathbb{Z}$; $\mathbb{Q}$; $[a, b]$; (a, b); $\mathbb{R}$.

(d) Is it true that a union of countable number of nowhere dense sets is nowhere dense?

(e) Is Cantor set nowhere dense?

($\star$) Is it possible to cover a closed interval by a countable number of nowhere dense sets?

Due Date: Friday, March 13, at the beginning of class.