Jacobs University
School of Engineering and Science
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Spring Term 2009

Problem Set 5

Analysis II

Homework Problems

Throughout this assignment, all the sets are subsets of metric space.

5.1. Which of the following properties of a map $f : X \to Y$ is equuivalent to continuity?

(a) Image of open set is open.

(b) Image of closed set is closed.

(c) Preimage of closed set is closed.

(d) For any $E \subset X$ holds $f(\overline{E}) \subset \overline{f(E)}$.

5.2. Show that a composition of continuous maps is continuous.

5.3. Let f be continuous. Show that

(a) image of connected set is connected;

(b) image of path-connected set is path-connected.

5.4. Which of the following maps $\mathbb{R}^2 \to \mathbb{R}^2$ are continuous? Which are homeomorphisms?

(a) $(x, y) \to (x + x^5, x^2 + y + e^{\sin x \cos x})$;

(b) $(x, y) \to (x^m + y^m, x^n + y^n)$, $n, m \in \mathbb{N}$;

(c) $(x, y) \to (x + y + \sin x, x - y + \cos y)$.

5.5. Suppose that sequences of real-valued functions (f_n) and (g_n) on $E \subset \mathbb{R}$ converge uniformly to f and g respectively.

(a) Is it true that there exist a uniform limit of sequences $(f_n + g_n)$; $(f_n g_n)$?

(b) Does the answer change if f_n, g_n are bounded?

(c) Does the answer change if E is a closed interval?

5.6. Two metrics ρ_1 and ρ_2 on X are called *equivalent* (notation: $\rho_1 \sim \rho_2$) if the identity map $\mathrm{id} : X \to X$ is a homeomorphism.

(a) Show that $\rho_1 \sim \rho_2$ if and only if any open set in any of these metrics is open in the other one.

(b) Let $f : X \to Y$ be continuous. Show that f will remain continuous if we substitute metrics in X and Y by equivalent ones.

(c) Are the following metrics on $\mathbb{R}^n$ equivalent?

$$\rho_1(x, y) = \max_{i \le n} |x_i - y_i|; \qquad \rho_2(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

5.7. ($\star$) Let $A, B \subset X$ be disjoint closed sets. Show that there exists a continuous function $f : X \to \mathbb{R}$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

5.8. ($\star$) Show that $K \subset X$ is compact if and only if any of the following holds:

(a) every continuous function $f : K \to \mathbb{R}$ is bounded;

(b) for any continuous function $f : K \to \mathbb{R}$ the image of K contains $\sup_{x \in K} f(x)$.

Due Date: Friday, March 20, at the beginning of class.