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Spring Term 2009

Problem Set 6

Analysis II

Homework Problems

Throughout this assignment, all the sets are subsets of metric space.

6.1. Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of continuous functions. Suppose that $\{f_n\}$ converges uniformly on (a, b). Show that $\{f_n\}$ converges uniformly on $[a, b]$.

6.2. Let m be positive integer. Define function

$$f_m(x) = \lim_{n\to\infty} (\cos m!\pi x)^{2n}$$

Show that $\{f_m\}$ converges and find the limit. Is the convergence uniform on $\mathbb{R}$? On $[0, 1]$?

6.3. Let $\{f_n\}$ be a sequence of continuous functions converging uniformly to f on E, and let $\{x_n\}$ be a sequence of points of E converging to x. Show that

$$\lim_{n\to\infty} f_n(x_n) = f(x)$$

6.4. *Series of functions* Let $f_n : X \to \mathbb{C}$ be a sequence of functions. A sum $\sum\limits_{n=1}^{\infty} f_n(x)$ is called *convergent* on $E \subset X$ if the sequence $s_m(x) = \sum\limits_{n=1}^{m} f_n(x)$ converges on E. $\sum\limits_{n=1}^{\infty} f_n(x)$ is called *uniformly convergent* on E if $\{s_m\}$ converges on E uniformly.

(a) *Cauchy criterion*

A sum $\sum\limits_{n=1}^{\infty} f_n(x)$ converges uniformly if and only if

$$\forall\, \varepsilon > 0\, \exists\, N \text{ such that } \forall\, k, m > N \ \left| \sum_{n=k}^{m} f_n(x) \right| < \varepsilon$$

(b) *Weierstrass test*

Suppose that $|f_n(x)| \le M_n$ on E, and the sum $\sum\limits_{n=1}^{\infty} M_n$ converges. Then $\sum\limits_{n=1}^{\infty} f_n(x)$ converges uniformly on E.

(c) Suppose that $|f_n(x)| \le g_n(x)$ on E, and the sum $\sum\limits_{n=1}^{\infty} g_n(x)$ converges uniformly on E. Then $\sum\limits_{n=1}^{\infty} f_n(x)$ converges uniformly on E.

(d) Let $\sum\limits_{n=1}^{\infty} f_n(x)$ converge uniformly on E. Then the sequence $\{f_n(x)\}$ converges uniformly to 0.

(e) Let $\sum\limits_{n=0}^{\infty} c_n(z - z_0)^n$ be a series, $z, z_0 \in \mathbb{C}$. Show that if $\sum\limits_{n=0}^{\infty} c_n(z - z_0)^n$ converges at point $\xi \ne z_0$, then it converges uniformly in any circle $\{z \in \mathbb{C} : |z - z_0| < q|\xi - z_0|\}$, $q < 1$.

6.5. *Abel-Dirichle test*

A set of functions $\mathcal{M}$ is called *uniformly bounded* on E if there exists M such that $|f(x)| < M$ for any $f \in \mathcal{M}$ and $x \in E$.

(a) *Abel transformation*

Show that

$$\sum_{k=n}^{m} a_k b_k = A_m b_m - A_{n-1} b_n + \sum_{k=n}^{m-1} A_k (b_k - b_{k+1})$$

where $a_k = A_k - A_{k-1}$.

(b) Suppose that $\{b_n\}$ is monotonic. Show that

$$\sum_{k=n}^{m} a_k b_k \leq 4 \max_{n-1 \leq k \leq m} |A_k| \max\{|b_n|, |b_m|\}$$

(c) *Dirichle test for uniform convergence of series*

Suppose that partial sums $s_m(x) = \sum_{n=1}^{m} f_n(x)$ of series $\sum_{n=1}^{\infty} f_n(x)$ are uniformly bounded on E, and sequence $\{g_n(x)\}$ is monotonic and uniformly converges to 0 on E. Then the series $\sum_{n=1}^{\infty} f_n(x) g_n(x)$ converges uniformly on E.

(d) *Abel test for uniform convergence of series*

Suppose that series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on E, and sequence $\{g_n(x)\}$ is monotonic and uniformly bounded on E. Then the series $\sum_{n=1}^{\infty} f_n(x) g_n(x)$ converges uniformly on E.

6.6. Show that the following series converges on $\mathbb{R}$. Is the convergence uniform?

$$\text{(a)} \quad \sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n} \qquad\qquad \text{(b)} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^2}{(1+x^2)^n}$$

6.7. ($\star$) Does there exist a continuous function on $\mathbb{R}$ which is not monotonic on any interval?

Due Date: Friday, March 27, at the beginning of class.