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School of Engineering and Science
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Spring Term 2009

Problem Set 7

Analysis II

Homework Problems

7.1. Let $A \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ be invertible, denote the norm of A by λ. What are the possible values of the norm of A^{-1}?

7.2. Let $f(x,y) = \frac{xy}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Prove that $(D_1 f)(x,y), (D_2 f)(x,y)$ exist at every point $(x,y) \in \mathbb{R}^2$, however f is not continuous at $(0,0)$.

7.3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \sqrt{|xy|}$. Show that f is not differentiable at the point $(0,0)$.

7.4. Let $E \subset \mathbb{R}^n$ be a connected open set, and let $f, g : E \to \mathbb{R}$ be differentiable.

(a) Show that fg is differentiable, and $(fg)'(x) = f(x)g'(x) + f'(x)g(x)$.

(b) Suppose that $g(x) \neq 0$ for some $x \in E$. Show that f/g is differentiable in x, and

$$\left(\frac{f}{g}\right)'(x) = \frac{1}{g^2(x)}(g(x)f'(x) - f(x)g'(x))$$

7.5. (a) Let $x, y \in \mathbb{R}^m$. Define a map $\gamma : [0,1] \to \mathbb{R}^m$, $\gamma(t) = (1-t)x + ty$. Show that γ is differentiable, and find its derivative.

(b) Let $E \subset \mathbb{R}^n$ be a connected open set, and let $f : E \to \mathbb{R}^m$ be differentiable. Show that if $f'(x) = 0$ for all $x \in E$ then f is constant.

Hint: Prove the statement for a ball, and then use the fact that a connected open set in $\mathbb{R}^n$ is path-connected.

7.6. Suppose $E \subset \mathbb{R}^n$ is open, and let $f : E \to \mathbb{R}$ is differentiable and has a local maximum in $x \in E$. Show that gradient of f at x is zero.

Due Date: Monday, April 6.