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Spring Term 2009

Problem Set 8

Analysis II

Homework Problems

8.1. Show that function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = t + 2t^2 \sin \frac{1}{t}, \quad f(0) = 0$$

satisfies all the assumptions of the inverse function theorem except continuity of f', and the inverse function theorem fails for f.

8.2. In which points the function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (x^2 - y^2, 2xy)$ is locally invertible? Is f globally invertible?

8.3. Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x, y_1, y_2) = x^2 y_1 + e^x + y_2$. Show that there is an open neighborhood $U \subset \mathbb{R}^2$ of $(1, -1)$ and a differentiable function $g : U \to \mathbb{R}$ such that $g(1, -1) = 0$ and $f(g(y_1, y_2), y_1, y_2) = 0$. Compute the derivative of g at $(1, -1)$.

8.4. Define function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}, \qquad f(0, 0) = 0$$

Show that

(a) f and its partial derivatives are continuous in $\mathbb{R}^2$;

(b) mixed second order derivatives exist at every point of $\mathbb{R}^2$, continuous in $\mathbb{R}^2 \setminus \{(0, 0)\}$, but do not coincide at $(0, 0)$.

8.5. Find local extrema of the following functions:

(a) $f(x, y) = 2x^3 - 3x^2 + 2y^3 + 3y^2$;

(b) $f(x, y) = x^4 + y^4 - 2x^2$.

8.6. Let $f : E \to \mathbb{R}$, $E \subset \mathbb{R}^n$, $f \in \mathcal{C}^{(m)}(E)$. Suppose that $[x, x + h] \subset E$. Use Taylor series of function $\varphi(t) = f(x + th)$ to show that

$$f(x + h) = f(x) + \sum_{k=1}^{m-1} (h_1 D_1 + \cdots + h_n D_n)^k f(x) + r(h), \qquad \lim_{h \to 0} \frac{r(h)}{|h|^{m-1}} = 0$$

8.7. ($\star$) Let $a \in \mathbb{R}^n$, $f : \overline{N_\varepsilon(a)} \to \mathbb{R}$ be a continuous function, and suppose that f is differentiable in $N_\varepsilon(a)$. Suppose also that f vanishes on the boundary of $\overline{N_\varepsilon(a)}$. Show that there exists $b \in N_\varepsilon(a)$ such that $\nabla f(b) = 0$.

Due Date: Monday, April 20.