Jacobs University
School of Engineering and Science
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Spring Term 2009

Problem Set 9

Analysis II

Homework Problems

A set $A \subset \mathbb{R}$ has *(Lebesgue) measure zero* (or *is zero set*) if for any positive ε there exists a countable set of intervals covering A such that sum of lenghts of any finite subset does not exceed ε.

9.1. Show that

(a) any subset of zero set is zero set;

(b) countable union of zero sets is zero set;

(c) Cantor set is zero set;

9.2. (a) Is it true that zero set is nowhere dense?

($\star$) Is it true that nowhere dense set is zero set?

9.3. Show that a closed (or open) interval is not a zero set.

9.4. Prove that topology in $\mathbb{R}^n$ has *countable base*, i.e. there exist a countable number of open sets $\{U_i\}$, such that any open set $U \subset \mathbb{R}^n$ is a countable union of some subcollection of $\{U_i\}$.

In the sequel all the sets are subsets of a unit square in $\mathbb{R}^2$.

9.5. Show that

(a) any open set is measurable;

(b) any closed set is measurable;

(c) any set obtained by countable number of operations of intersection, union and taking complement from open sets is measurable. Such sets are called *Borel sets*.

9.6. Let A be a measurable set. Then for any positive ε there exist open G and closed F, such that

$$F \subset A \subset G, \qquad \text{and} \quad \mu(G \setminus F) < \varepsilon$$

Due Date: Monday, May 4.