Jacobs University
School of Engineering and Science

Spring Term 2010

Linear Algebra II, Homework 1

Due Date: Wednesday, February 10, in class.

Problems marked $(\star)$ are bonus ones.

1.1. Which of the following maps $g : \mathrm{Mat}_n(\mathbb{C}) \times \mathrm{Mat}_n(\mathbb{C}) \to \mathbb{C}$ are inner products? Which of them are orthogonal (symplectic, Hermitian), non-degenerate?

(a) $g(A, B) = \mathrm{tr}\ AB$;

(b) $g(A, B) = \det AB$;

(c) $g(A, B) = \mathrm{tr}\ AB^t$;

(d) $g(A, B) = \mathrm{tr}\ A\overline{B^t}$.

1.2. Let (L, g) be an inner product space. Show that $g(x, y) = 0$ implies $g(y, x) = 0$ for all $x, y \in L$ if and only if g is either orthogonal, or symplectic, or Hermitian, or skew-Hermitian (i.e., $g(x, y) = -\overline{g(y, x)}$).

1.3. Let (L, g) be a finite-dimensional inner product space, g is non-degenerate, $\dim L = n$. Show that Gram matrix of vectors $\{v_1, \dots, v_n\}$ is non-degenerate if and only if $\{v_1, \dots, v_n\}$ are linearly independent.

1.4. Let (L, g) be an inner product space, g is non-degenerate.

(a) Prove that if L is finite-dimensional, then for any $f \in L^*$ there exists $l \in L$ such that for any $v \in L$ $\ f(v) = g(l, v)$.

(b)$(\star)$ Show that if L is infinite-dimensional then (a) may not hold.
Hint. Consider an inner product $g(p, q) = \int_{-1}^{1} pq$ on the space of all polynomials with real coefficients on $[-1, 1]$, and $f(p) = p(0)$.

1.5. Let g be an inner product defined on a linear space L. For a subspace L_0 of L define $L_0^\perp = \{v \in L |\ g(v, x) = 0\ \forall x \in L_0\}$.

(a) Show that $L_0 \subseteq (L_0^\perp)^\perp$ for any subspace L_0;

(b) prove that if L_1 and L_2 are subspaces of L then $(L_1 + L_2)^\perp = L_1^\perp \cap L_2^\perp$;

(c) find $L_0^\perp$ if $L = \mathrm{Mat}_n(\mathbb{R})$, L_0 is the subspace of diagonal matrices, and $g(A, B) = \mathrm{tr}\ AB^t$.