Jacobs University
School of Engineering and Science

Spring Term 2010

Linear Algebra II, Homework 10

Due Date: Wednesday, May 12.

Problems marked $(\star)$ are bonus ones.

10.1. Let $f \in \mathrm{End}(L)$, $g \in \mathrm{End}(M)$, and dimensions of L and M are n and m respectively. Define a linear operator $f \otimes g \in \mathrm{End}(L \otimes M)$ on factorizable tensors by

$$(f \otimes g)(l \otimes m) = f(l) \otimes g(m)$$

for any $l \in L, m \in M$. Show that

(a) $f \otimes g$ is well defined; (b) $\mathrm{tr}\ f \otimes g = \mathrm{tr}\ f\ \mathrm{tr}\ g$; (c)$(\star)$ $\det f \otimes g = (\det f)^m (\det g)^n$.

10.2. Let $\{e_1, \dots, e_n\}$ be a basis of L, and $\{e^1, \dots, e^n\}$ be a dual basis of L^*. Compute the contraction of the tensor

(a) $T = e^j \otimes e_i$; (b) $T = \sum_{j=1}^{n} e^j \otimes e_{n-j}$; (c) $T = \sum_{i,j=1}^{n} (-1)^{i+j} e^j \otimes e_i$.

10.3. Let $\{e_1, \dots, e_n\}$ be a basis of L. Show that symmetric tensors of the form

$$e_1^{a_1} \dots e_n^{a_n} \ (= S(\underbrace{e_1 \otimes \cdots \otimes e_1}_{a_1} \otimes \underbrace{e_2 \otimes \cdots \otimes e_2}_{a_2} \otimes \cdots \otimes \underbrace{e_n \otimes \cdots \otimes e_n}_{a_n})), \qquad a_1 + \cdots + a_n = q$$

span the symmetric power $S^q(L)$.

10.4. $(\star)$ Let $f \in \mathrm{End}(L)$, $\{e_1, \dots, e_n\}$ is a basis of L. Define a *symmetric square* $S^2(f) \in \mathrm{End}(S^2(L))$ by

$$(S^2(f))(e_i e_j) = S(f(e_i) \otimes f(e_j))$$

(a) Show that $S^2(f)$ is well defined.

(b) Show that $\mathrm{tr}\ S^2(f) = \frac{1}{2}((\mathrm{tr}\ f)^2 + \mathrm{tr}\ f^2)$.

(c) Suppose that the characteristic polynomial of f has roots $\lambda_1, \dots, \lambda_n$. Show that the characteristic polynomial of $S^2(f)$ has $n(n+1)/2$ roots $\lambda_i \lambda_j$, $1 \le i \le j \le n$.