Jacobs University
School of Engineering and Science
Spring Term 2010

Linear Algebra II, Homework 6

Due Date: Thursday, March 25, in class.

Problems marked ($\star$) are bonus ones.

6.1. Let L be a unitary space. For every self-adjoint operator f define an inner product

$$g_f(x,y) = (f(x), y)$$

Show that g_f is a well-defined Hermitian inner product, and the map $f \to g_f$ is a bijection between the set of self-adjoint operators and Hermitian forms.

6.2. Let f^* be an operator adjoint to f in unitary space L, i.e.

$$(f^*(x), y) = (x, f(y))$$

for all $x, y \in L$. Is it true that $\operatorname{im} f^* = (\ker f)^\perp$, and $\ker f^* = (\operatorname{im} f)^\perp$?

6.3. Show that an operator in Euclidean space is self-adjoint if and only if it is diagonalizable in orthogonal basis.

A self-adjoint operator f is *non-negative* if $(f(x), x) \geq 0$ for every x, and *positive* if $(f(x), x) > 0$ for every $x \neq 0$.

6.4. (a) Given any operator f on a unitary space, show that f^*f is a non-negative self-adjoint operator, and it is positive if and only if f is invertible.

(b) Show that for any non-negative self-adjoint operator f there exists non-negative self-adjoint operator h such that $f = h^2$.

6.5. ($\star$) Let f_1, f_2 be positive self-adjoint operators. Show that

(a) if h is self-adjoint, $h^2 = f_1$, and f_1 and f_2 commute, then h and f_2 commute;

(b) $f_1 f_2$ is positive self-adjoint if and only if f_1 and f_2 commute.