Jacobs University
School of Engineering and Science

Spring Term 2010

Linear Algebra II, Homework 7

Due Date: Thursday, April 8, in class.

Problems marked ($\star$) are bonus ones.

7.1. Let A be a real square matrix. Show that A is symmetric if and only if there exist real invertible matrix C and real diagonal matrix D with

$$A = C^{-1}DC$$

7.2. Let f be a normal operator in a unitary space L. Show that

(a) kernels of f and f^* coincide;

(b) $\operatorname{im} f = \ker f^*$;

(c) images of f and f^* coincide;

(d) L is a direct orthogonal sum of the kernel and image of f.

7.3. Show that an operator f in a unitary space is normal if and only if every eigenvector of f is an eigenvector of f^*.

7.4. Let L be an orthogonal space with form

$$(x, y) = -x_0y_0 + \sum_{i=1}^{n} x_iy_i$$

Let $(x, x) < 0$ and $(y, y) < 0$. Show that $(x, y) < 0$ if and only if $x_0y_0 > 0$.

7.5. ($\star$) **(Polar decomposition)**

Let f be an invertible operator in a unitary space. Define r_1, r_2 to be positive self-adjoint operators, such that $r_1^2 = ff^*$, $r_2^2 = f^*f$ (see Problem 6.4b).

(a) Show that there exist unitary operators u_1, u_2, such that

$$f = r_1u_1 = u_2r_2$$

The representations above are called *polar decompositions* of f.

(b) Show that polar decompositions $f = r_1u_1 = u_2r_2$ are unique.