Sector Decomposition

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Outline

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- A method of evaluating parameter integrals that occur in perturbative QFT
- Can be used to calculate virtual and real corrections to processes at higher orders

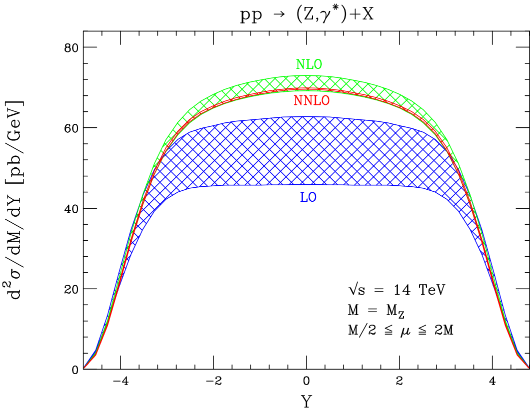
Outline

- Current experimental accuracy 1%
- Future precision experiments will require theoretical predictions at 0.1%
- Computation of higher order corrections is vital to achieve this level of accuracy

μ Dependence (I)

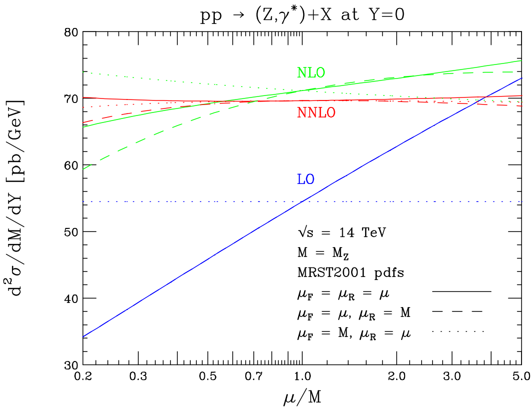
- Evaluation of these high order corrections are formally infinite, so we use dimensional regularisation ($D = 4 - 2\epsilon$) to describe these infinities. This introduces an energy scale μ_R
- Processes with partonic initial states are factorized so that above a certain energy scale μ_F, partonic interactions ($gg, qg, q\bar{q}$...) are treated separately from the parton distribution function
- These μ_R, μ_F are put in by hand, and thus the true result should have no μ dependence (conventionally $\mu_R = \mu_F = \mu$)

μ Dependence (II)



Anastasiou, Dixon, Melnikov and Petriello, hep-ph/0312266

μ Dependence (III)



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1. Introduction
   - What is Sector Decomposition?
   - Why is Sector Decomposition Important?
2. The Algorithm Explained
   - Goal
   - Method
3. Future Work
   - Linear Divergences
   - Phase Space
   - $t\bar{t}$ Production

Getting From This...

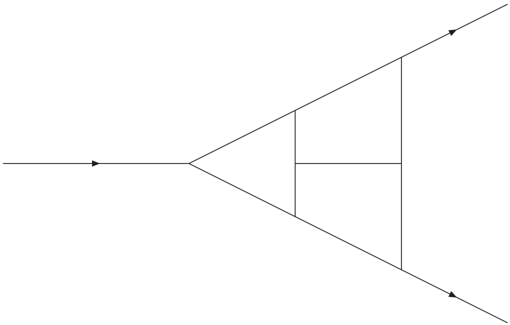

Figure: $A_{9,1}$ Massless Three-Loop Form Factor

To This

$$A_{9,1} = i\Gamma(3+3\epsilon)(-q^2 - i\eta)^{-3-3\epsilon}(-0.027872/\epsilon^5 + 0.374876/\epsilon^4 - 3.492757/\epsilon^3 + 21.367526/\epsilon^2 - 104.122985/\epsilon + 353.981135 + O(\epsilon))$$

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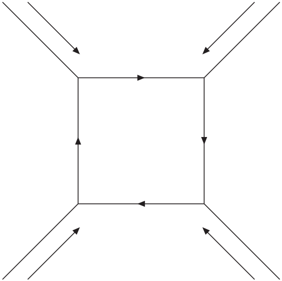
Feynman Parameters

- Write down amplitude using Feynman rules
- Use Feynman parameters and integrate over loop momenta
- What is left is $I = \int_0^1 (\prod_{j=1}^N dx_j)\delta(1 - \sum_{i=1}^N x_i)\frac{U(\mathbf{x})^{a+b\epsilon}}{F(\mathbf{x})^{c+d\epsilon}}$
- U is a function of **x**, and F is a function of **x** and external invariants (s,m^2,...), and have zeroes when all or some of $x_i \to 0$

Example

- As an example, I will consider the massless 1-loop box



Example

- $N = 4$
- $I \sim \int d^D k \frac{1}{k^2(k+p_1)^2(k+p_1+p_3)^2(k-p_2)^2}$
- $$\int d^D k \int_0^1 d^4 x \frac{\delta(1-\sum_{i=1}^4 x_i)}{(k^2(x_1+x_2+x_3+x_4)+2(x_2 p_1+x_3 p_1-x_4 p_2+x_3 p_3)\cdot k+2x_3 p_1\cdot p_3)^4}$$
- $\sim \int_0^1 d^4 x \frac{\delta(1-\sum_{i=1}^4 x_i)(x_1+x_2+x_3+x_4)^{2\epsilon}}{(-s_{12}x_1x_3-s_{13}x_2x_4)^{2+\epsilon}}$

Primary Decomposition

- Split $I = \sum_{k=1}^{N} I_k$, where I_k is restricted to the region $x_k > x_i \forall i$
- Rescaling, Relabelling and integrating out δ function wrt x_k gives $I_k = \int_0^1 (\prod_{j=1}^{N-1} dt_j) \frac{\tilde{U}(\mathbf{t})^{a+b\epsilon}}{\tilde{F}(\mathbf{t})^{c+d\epsilon}}$
- $\tilde{U}$ and $\tilde{F}$ typically still have zeroes as some subset of $t_i \to 0$

Example

- Consider I_4:
- $x_1 = x_4 t_1, x_2 = x_4 t_2, x_3 = x_4 t_3, x_4 = x_4$
- $I_4 = \int_0^1 d^3t \frac{(1+t_1+t_2+t_3)^{2\epsilon}}{(-s_{12}t_1t_3 - s_{13}t_2)^{2+\epsilon}} \int_0^1 dx_4 \frac{\delta(1-x_4(1+t_1+t_2+t_3))}{x_4}$
- $I_4 = \int_0^1 d^3t \frac{(1+t_1+t_2+t_3)^{2\epsilon}}{(-s_{12}t_1t_3 - s_{13}t_2)^{2+\epsilon}}$

Iterated Decomposition

- Search for a subset of the $\{t_{i_1}, ... t_{i_p}\}$ such that at least one of $\tilde{U}, \tilde{F} \to 0$ as $\{t_{i_1}, ... t_{i_p}\} \to 0$
- If no such subset exists then the iteration terminates
- Else $I_k = \sum_{q=1}^{p} I_{k,q}$, where $I_{k,q}$ has $t_{i,q} > t_{i,r} \forall r$
- Rescale $\{t_{i_1}, ... t_{i_p}\}$ and factor $t_{i,q}$ out of $\tilde{U}, \tilde{F}$ where possible.
- Repeat for each new subsector created.

Example

- Consider $I_4 = \int_0^1 d^3t \frac{(1+t_1+t_2+t_3)^{2\epsilon}}{(-s_{12}t_1t_3 - s_{13}t_2)^{2+\epsilon}}$:
- Numerator is already finite as $\mathbf{t} \to 0$. Denominator $\to 0$ as t_1 and t_2 both $\to 0$
- Consider $I_{4,2}$ (ie $t_2 > t_1$): $t_1 = t_2' t_1'$, $t_2 = t_2'$
- $I_{4,2} = \int_0^1 dt_1' dt_2' dt_3 t_2'^{-1-\epsilon} \frac{(1+t_1't_2'+t_2'+t_3)^{2\epsilon}}{(-s_{12}t_1't_3 - s_{13})^{2+\epsilon}}$
- $I_{4,2} = \int_0^1 t_2^{-1-\epsilon} d^3t \frac{(1+t_1t_2+t_2+t_3)^{2\epsilon}}{(-s_{12}t_1t_3 - s_{13})^{2+\epsilon}}$

Subtraction (I)

- After the iteration terminates and the subsectors are relabelled we have $I = \sum_{m=1}^{\#subsectors} I_m$
- Each I_m is of the form $\int_0^1 (\prod_{j=1}^{N-1} dt_j t_j^{e_j+f_j\epsilon}) \frac{\tilde{U}(\mathbf{t})^{a+b\epsilon}}{\tilde{F}(\mathbf{t})^{c+d\epsilon}}$
- $\tilde{U}$ and $\tilde{F}$ are $O(1)$ at $\mathbf{t} \to 0$, so rewrite $\frac{\tilde{U}(\mathbf{t})^{a+b\epsilon}}{\tilde{F}(\mathbf{t})^{c+d\epsilon}} \equiv g(\mathbf{t}, \epsilon) = O(1) + ...$

Subtraction (II)

- All the singularities are contained in the $\prod_{j=1}^{N-1} dt_j t_j^{e_j+f_j\epsilon}$
- If $e_j > -1$ then there is no singularity in t_j
- If $e_j = -1$, subtraction is needed
- Write $g(\mathbf{t},\epsilon) \equiv g(t_j = 0,\epsilon) + (g(\mathbf{t},\epsilon) - g(t_j = 0,\epsilon))$
- $\int_0^1 t^{-1+f\epsilon} g(0,\epsilon) dt = \frac{g(0,\epsilon)}{f\epsilon} \int_0^1 dt$
- $\int_0^1 t^{-1+f\epsilon} (g(t,\epsilon) - g(0,\epsilon)) = O(1)$
- If $e_j <= -2$ then the procedure still works, but with more terms of the Taylor expansion included

Example

- Consider $I_{4,2} = \int_0^1 t_2^{-1-\epsilon} d^3t \frac{(1+t_1 t_2+t_2+t_3)^{2\epsilon}}{(-s_{12} t_1 t_3 - s_{13})^{2+\epsilon}}$
- $I_{4,2} = \int_0^1 dt_1 dt_3 (\frac{(1+t_3)^{2\epsilon}}{(-s_{12} t_1 t_3 - s_{13})^{2+\epsilon}} \int_0^1 dt_2 (t_2^{-1-\epsilon})$
$+\int_0^1 dt_2 (t_2^{-1-\epsilon} (\frac{(1+t_1 t_2+t_2+t_3)^{2\epsilon}}{(-s_{12} t_1 t_3 - s_{13})^{2+\epsilon}} - \frac{(1+t_3)^{2\epsilon}}{(-s_{12} t_1 t_3 - s_{13})^{2+\epsilon}})))$
- $= \int_0^1 d^3t ((\frac{-1}{\epsilon} \frac{(1+t_3)^{2\epsilon}}{(-s_{12} t_1 t_3 - s_{13})^{2+\epsilon}})$
$+ t_2^{-1-\epsilon} (\frac{(1+t_1 t_2+t_2+t_3)^{2\epsilon}}{(-s_{12} t_1 t_3 - s_{13})^{2+\epsilon}} - \frac{(1+t_3)^{2\epsilon}}{(-s_{12} t_1 t_3 - s_{13})^{2+\epsilon}}))$

Numerical Integration

- $I(\epsilon) = \sum_m I_m(\epsilon)$
- Perform the Laurent Expansion in ϵ
- For each order of ϵ the coefficient is a sum of well-behaved integrals over the $N-1$ dimensional unit hypercube, each of which can be calculated via Monte Carlo integration to yield the full result

Example

- For ease of notation I shall set $s_{12} = s_{13} = -1$
- $I_{4,2} = \int_0^1 d^3t((\frac{-1}{\epsilon}\frac{(1+t_3)^{2\epsilon}}{(1+t_1t_3)^{2+\epsilon}})$

$+ \mathrm{t}_2^{-1-\epsilon}(\frac{(1+t_1t_2+t_2+t_3)^{2\epsilon}}{(1+t_1t_3)^{2+\epsilon}} - \frac{(1+t_3)^{2\epsilon}}{(1+t_1t_3)^{2+\epsilon}}))$

- $I_{4,2} = \frac{-1}{\epsilon}\int_0^1 d^3t\frac{1}{(1+t_1t_3)^2} + \int_0^1 d^3t\frac{2log(1+t_3)-log(1+t_1t_3)}{(1+t_1t_3)^2} + O(\epsilon)$
- $= \frac{-log(2)}{\epsilon} + \frac{\pi^2+6log(2)^2-3log(16)}{12} + O(\epsilon)$
- Full numerical result is $\frac{4}{\epsilon^2} - \frac{4}{\epsilon} - 12.449 + O(\epsilon)$

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- Complicated loop diagrams yield a lot of variables with $t^{-2+f\epsilon}$

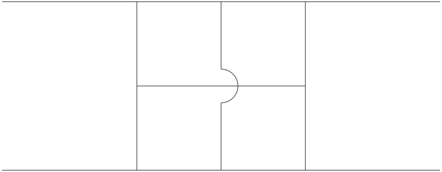

Figure: Four-Point Three-Loop Diagram

- These divergences rapidly increase computation time for the subtraction, and in many cases the numerical integration becomes unworkable, as eg.
  $\frac{1-log(1+t)-\frac{1}{1+t}}{t^2} \rightarrow \frac{-1}{2}$ as $t \rightarrow 0$
  but this behaviour is not seen by the numerical integration
- Taylor Expansions in these variables provide one way around the problem, but this vastly increases both the time and memory required to complete the calculation. For more than 2 of these poles, this method is prohibitively expensive
- Test new methods to overcome this problem

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- Apply the method to real unresolved radiation
- Divergences can come from soft/collinear massless particles

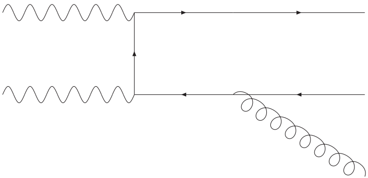
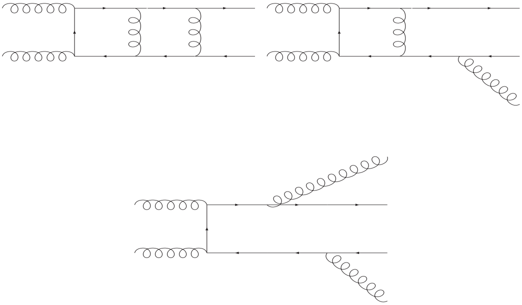


Figure: $\gamma\gamma \to q\bar{q}g$

Outline

- We aim to produce the full NNLO cross-section for $t\bar{t}$ production at the LHC, including two-loop, 1-loop $\times$ real radiation and double real radiation



Further Reading

For an indepth explaination of the method:
'Sector Decomposition'
Gudrun Heinrich
Arxiv:0803.4177