

# Black Holes, Vortices and Thermodynamics

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- Black Hole Thermodynamics
- Vortices
- Black Holes with Vortices
- Solving the Hierarchy Problem??

# Black Hole Thermodynamics

Hawking and Bekenstein in the early 70's conjectured that black holes have thermodynamic properties.

- Black holes have entropy  $S$ .
- Black holes have Hawking temperature  $T_H$ , consistent with thermodynamic relation between energy, entropy and temperature.

## Thermodynamics

- $S = \frac{\mathcal{A}}{4}$  where  $\mathcal{A}$  is the area of the event horizon.
- $T_H = \frac{\kappa}{2\pi}$  where  $\kappa$  is the surface gravity of the black hole.

# Path Integral Formulation

- In 1976 Hawking and Gibbons demonstrated that these thermodynamic results could be attained via a path integral approach to quantum gravity.
- In this approach one considers expressions of the form

$$\langle \Phi_2, t_2 | \Phi_1, t_1 \rangle = \langle \Phi_1 | e^{-iH(t_2-t_1)} | \Phi_2 \rangle = \int \mathcal{D}[\Phi] e^{iS[\Phi]}$$

if  $t_2 - t_1 = i\beta$  and  $\Phi_1 = \Phi_2$  then

$$\langle \Phi, t_2 | \Phi, t_1 \rangle = \text{Tr} e^{-\beta H} = Z = \int \mathcal{D}[\Phi] e^{iS[\Phi]}$$

- Here  $Z$  is the partition function of an ensemble at temperature  $T = \beta^{-1}$ ,  $\mathcal{D}[\Phi]$  is the measure of the space of all fields.

# Euclideanisation and Temperature

- In quantum gravity this is the same

$$Z = \int d[g]d[\phi]e^{iS[g][\phi]}$$

- For ease of calculation the metric is Euclideanised i.e.  $t \rightarrow i\tau$  and the metric becomes positive definite.
- Now, a periodicity can be associated with  $\tau$  meaning  $\beta = \Delta\tau$ .
- Then, by including all metrics that are asymptotically flat and have periodicity of the imaginary time coordinate  $\beta = \Delta\tau$ , the path integral gives the partition function for a system at temperature  $T = \beta^{-1}$ .

## Euclidean Schwarzschild black hole metric

$$ds^2 = \left(1 - \frac{r_s}{r}\right) d\tau^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

- Singular at  $r = r_s$ .
- A change variables of  $\rho^2 = (2r_s)^2\left(1 - \frac{r_s}{r}\right)$ , gives

$$ds^2 = \rho^2 \left(\frac{\tau}{2r_s}\right)^2 + d\rho^2 + r_s^2 d\Omega_{II}^2 \quad \text{as} \quad r \rightarrow r_s.$$

- $\rightarrow \tau$  must be periodic with period  $\beta = 4\pi r_s$
- Singularity is coordinate singularity
- Metric is only defined on  $r_s \leq r < \infty$
- The metric has topology  $\mathbb{R}^2 \times S^2$ .

# Black Hole Partition function

- For black holes the key contributions to the path integral come from geometries that have topology such as this i.e.  $\mathbb{R}^2 \times S^2$ .
- For the Schwarzschild black hole including this geometry alone results in the partition function from which the famous results can be derived.

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- A vortex is a non-perturbative, non-trivial solution of the field equations.
- This talk will consider only local Abelian Higgs vortices.
- They can be created during phase transitions.

## Abelian Higgs Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{D}^\mu\phi(\mathcal{D}_\mu\phi)^* - \frac{\lambda}{4}(\phi\phi^* - \eta^2)^2,$$

$$\phi(x)' = e^{i.e.\Lambda(x)}\phi(x), \quad A_\mu(x)' = A_\mu(x) - \partial_\mu\Lambda(x),$$

$$\mathcal{D}_\mu\phi = \partial_\mu\phi + ieA_\mu\phi.$$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{D}^\mu\phi(\mathcal{D}_\mu\phi)^* - \frac{\lambda}{4}(\phi\phi^* - \eta^2)^2$$

- If  $\eta > 0$  then it is the symmetry breaking scale, an energy scale below which  $\phi(x)$  acquires a  $\text{vev} \neq 0$ , the symmetry breaks and the theory undergoes a phase transition.
- It is likely that during a transition a non-trivial winding of the phase will appear about some point.
- For this winding to be reconciled at the origin,  $\phi$  must rise up the potential barrier to  $\phi = 0$ , thus a stable, localised, non-zero energy density appears which forms the vortex core.

- Finite energy considerations imply that  $\phi \rightarrow \eta$  as  $r \rightarrow \infty$  (it's vacuum value) and  $A_\mu$  must asymptotically be a pure gauge rotation.

## Simplest Field Configuration for Vortices

$$\phi = \eta X(r) e^{ik\theta}, \quad \begin{cases} X(0) = 0, \\ X(r) \rightarrow 1, \quad r \rightarrow \infty. \end{cases}$$
$$A_\mu = \frac{1}{e} (P(r) - k) \partial_\mu \theta, \quad \begin{cases} P(0) = k, \\ P(r) \rightarrow 0, \quad r \rightarrow \infty. \end{cases}$$

- This form simplifies the field equations for variables  $X(r)$  and  $P(r)$ .

## Field Equations In Minkowski Background

$$X'' = \frac{-X'}{r} + \frac{P^2 X}{r^2} + \frac{\lambda \eta^2}{2} (X^2 - 1) X,$$
$$P'' = \frac{P'}{r} + 2e^2 \eta^2 X^2 P.$$

- These coupled, second order, ordinary differential equations can be solved numerically.

# Field Distributions

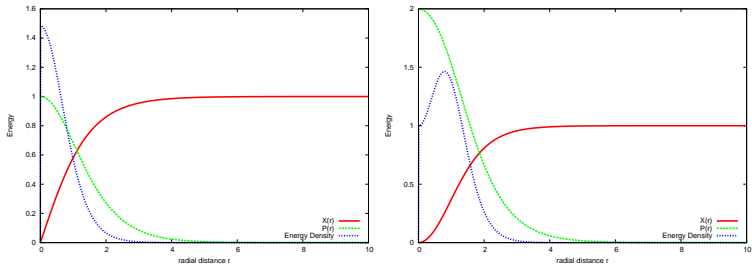


Figure: Field distribution for  $k=1$  and  $k=2$  vortices

# Vortices and Gravity

To include gravity:

- The Minkowski metric must be replaced by  $g_{\mu\nu}$ , the general metric.
- The field equations must now include components of the metric and be coupled to the Einstein equations. Giving more differential equations of more variables.

These equations have been solved for a vortex in an otherwise flat spacetime and give an interesting result.

- The geometry of the spacetime outside the core is locally identical to Minkowski but not globally.
- The effect of the vortex is to introduce a 'deficit angle' making the spacetime that of a snub-nosed cone.

$$\Delta = 8\pi G\mu$$

where  $\Delta$  is the deficit angle,  $G$  is Newtons constant and  $\mu$  is the vortex mass per unit length.

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# Vortex on a Black Hole

- The temperature of a black hole depends on the periodicity,  $\beta$ , of the imaginary temporal coordinate.
- The gravitational effect of a vortex on the surrounding space time is to reduce the period of the dimension in which its phase lies.
- Therefore, one might expect that a vortex on a black hole configured such that its phase lies in the temporal direction may effect the temperature of a black hole.



We now consider an Abelian Higgs Lagrangian with General Euclidean Schwarzschild metric

## Lagrangian and Metric

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \mathcal{D}^\mu \phi (\mathcal{D}_\mu \phi)^* + \frac{\lambda}{4} (\phi \phi^* - \eta^2)^2,$$
$$ds^2 = A^2 d\tau^2 + A^{-2} dr^2 + C^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

## Field Configuration

$$\phi = \eta X(r) e^{ik \frac{2\pi\tau}{\beta}},$$
$$A_\mu = \frac{2\pi}{\beta e} (P(r) - k) \partial_\mu \tau = \frac{2\pi}{\beta e} (P_\mu - k \partial_\mu).$$

This configuration ensures cylindrical symmetry about  $\tau$  which leads to  $A = A(r)$  and  $C = C(r)$ .

## Field Equations

Varying  $\phi$  and  $A_\mu$  gives

$$\frac{1}{C^2}(C^2 P')' = 2e^2 \eta^2 \frac{X^2 P}{A^2}$$
$$\frac{1}{C^2}(C^2 A^2 X')' = \frac{P^2 X 4\pi^2}{A^2 \beta^2} + \frac{\lambda \eta^2}{2} X(X^2 - 1).$$

Varying  $g^{\mu\nu}$  gives the Einstein equations, which for this case are:

$$C'' = 4\pi G \frac{C}{A^2} (T_0^0 - T_r^r)$$
$$((A^2)' C^2)' = 8\pi G C^2 (2T_\theta^\theta + T_r^r - T_0^0)$$
$$\frac{(A^2)' C'}{C} - \frac{1}{C^2} (1 - A^2 C'^2) = 8\pi G T_r^r$$

Where  $T_i^j$  are components of the energy-momentum tensor.

# Boundary conditions

These coupled, ordinary differential equations must be solved simultaneously along with the boundary conditions specified by finite energy constraints and regularity of the metric at the horizon.

## Boundary Conditions

$$C(r_s) = r_s$$

$$A(r_s) = 0$$

$$X(r_s) = 0$$

$$P(r_s) = 1$$

$$A(r_s)^2 = \frac{1}{r_s}$$

$$X(\infty) = 1$$

$$P(\infty) = 0.$$

- The problem complicated by the ‘mixed type’ boundary conditions.
- The method used involves a Runge-Kutta algorithm on the equations for the gravity fields and successive under-relaxation on the matter fields, repeated on successive iterations.

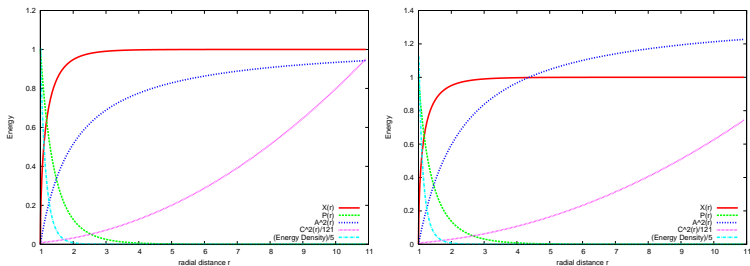


Figure: Field distribution for  $G=0.0$  and  $G=0.02$  vortices

- Caveat: There is a small numerical artefact in these solutions (not shown here) which needs some further investigation to ascertaining its origin.

# Results

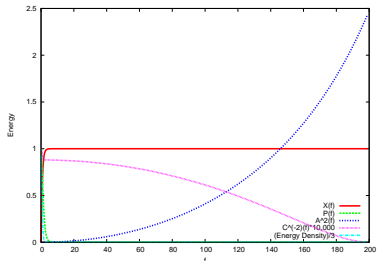
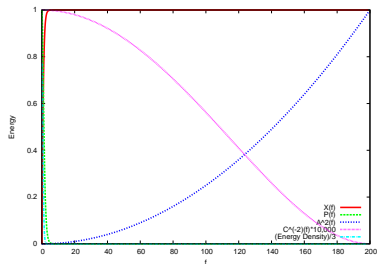
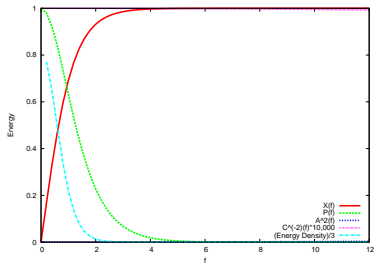


Figure: Field distribution for  $G=0.0$  (close up),  $G=0.0$  and  $G=0.02$  vortices (coordinates appear flat at the horizon).

Key observations:

- Gravity fields are asymptotically Schwarzschild.
- $A^2$ 's asymptotic value is increased by the presence of the gravitating vortex.

If we look at the asymptotic Schwarzschild where  $A^2$  has been multiplied by a constant  $\lambda$ .

$$ds^2 = \lambda^2 A^2 d\tau^2 + \frac{1}{\lambda^2} A^{-2} dr^2 + \frac{1}{\lambda^2} C^2 (d\theta^2 + \sin^2\theta d\phi^2).$$

This factor can only be absorbed by a rescaling of  $d\tau \rightarrow \frac{d\tau}{\lambda}$  and  $dr \rightarrow \lambda dr$

- Therefore the period at infinity,  $\tilde{\beta} = \frac{\Delta\tau}{\lambda} = \frac{\beta}{\lambda}$ , is reduced and the temperature of the black hole  $\tilde{T}_H = \frac{1}{\tilde{\beta}}$  is increased.

- Verified numerically that the presence of a vortex on a Euclidean Schwarzschild black hole increased the temperature of the system.
- This supports previous work of my supervisor and collaborators when looking analytically at the extreme case of a thin weakly gravitating vortex on a black hole.
- These results apply to the more general case of thicker and stronger gravitating vortices.
- This may well have important implications on other current work in the field.

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# Discrete Local Symmetry

- In the 70's the idea of discrete local symmetry was used as a way of endowing black holes with 'hair' despite the no hair theorems.

Consider a theory of two complex scalar fields  $\phi$  and  $\varphi$  invariant under a local  $U(1)$  gauge transformation.

$$\begin{aligned}\phi(x)' &= e^{ie\Lambda(x)}\phi \\ \varphi(x)' &= e^{\frac{ie\Lambda(x)}{N}}\varphi \\ A_\mu(x)' &= A_\mu(x) - \partial_\mu\Lambda(x),\end{aligned}$$

for some integer  $N$ . Now consider the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F^2 + (\mathcal{D}\phi)^2 + (\mathcal{D}\varphi)^2 - \frac{\lambda}{4}(\phi^2 - \eta^2)^2 + V(|\varphi|).$$

# Discrete Local Symmetry

- Suppose  $\eta$  is some energy scale at which  $\phi$  condenses and  $\varphi$  does not.
- $\phi$  will be invariant under additions of  $k\frac{2\pi}{e}$  to the phase whilst  $\varphi$  will not, being multiplied by the  $n$ th root of unity.
- The possible actions of these transformations generate a  $Z_N$  symmetry.

# Solving the Hierarchy Problem?

Gia Dvali (2007) postulated a novel solution to the hierarchy problem using black hole physics and the idea of discrete quantum charges.

- Consider  $N$  bosonic fields each of mass  $\lambda$  and charged under an individual  $Z_2$  symmetry as just described.
- Imagine a black hole constructed of one of each of these species of field. The mass of such a black hole will be

$$M_{BH} = N\lambda.$$

- The black hole will emit particles as Hawking radiation.
- These are gauge conserved charges and must be revealed after evaporation.

# Solving the Hierarchy Problem?

- The black hole will only emit particles of mass  $\lambda$  when  $T_H \approx \lambda$
- At this point the mass of the black hole is given by

$$\tilde{M}_{BH} = \frac{M_P^2}{T_H} \approx \frac{M_P^2}{\lambda}$$

- Only at this point can the black hole start to reveal the  $N$  particles. By conservation of energy it must reveal them all, thus

$$\begin{aligned}\tilde{M}_{BH} &= N\lambda \\ \rightarrow M_P^2 &\approx N\lambda^2.\end{aligned}$$

- Therefore the postulation of  $N = 10^{32}$  beyond the standard quantum field species will resolve the hierarchy problem.

- Symmetry breaking is required to form these discrete charged fields, and therefore vortices are permitted.
- The presence of vortices increases the temperature of a black hole; by an amount at least proportional to the number of vortices.

$$T_{NoVortex} = \frac{1}{8\pi GM} = \frac{M_P^2}{M_{BH}} \approx \lambda$$
$$T_{NVortices} = \frac{1}{8\pi GM}(N) = \frac{M_P^2}{M_{BH}}(N) \approx \lambda$$
$$\rightarrow M_P^2 \approx \frac{N\lambda^2}{(N)} = \lambda^2.$$

# Questions about Criticism

- What is the true significance of the discrete charge in relation to the path integral?
  - i.e. Does the presence of a vortex in the theory affect the temperature of the black hole irrespective of discrete charge in the black hole?
- Are  $N$  voracities truly necessary to have  $N$  different field species?
- To answer these questions we must look closely at how the discretely charged fields relate to the 'no hair' theorems.