Boundaries in 2d Integrable Quantum Field Theory

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Outline

- Integrability in 2d QFT
- Adding a Boundary
- The Boundary S-matrix
- The Ising model
- Ising boundary conditions
- Correlation functions
Integrability in 2 Dimensions

Usually use Euclidean time ($y = it$) and flat coordinates given by

$$z = x + iy \quad \bar{z} = x - iy$$

$$\partial = \partial_z = \frac{1}{2} (\partial_x - i\partial_y) \quad \bar{\partial} = \partial_{\bar{z}} = \frac{1}{2} (\partial_x + i\partial_y)$$

Can define a theory in this space via an action or a CFT perturbed by a relevant field. In both cases the symmetric stress tensor satisfies

$$\bar{\partial} T = \partial \Theta \quad \partial \bar{T} = \bar{\partial} \Theta$$

and Hamiltonian is given by

$$H = \int dx [T + \bar{T} + 2\Theta]$$
Integrals of Motion

Assuming our model is Integrable we have an infinite set of local fields satisfying

\[ \bar{\partial} T_{s+1} = \partial \Theta_{s-1} \quad \partial \bar{T}_{s+1} = \bar{\partial} \bar{\Theta}_{s-1} \]

\[ P_s = \int dx (T_{s+1} + \Theta_{s-1}) \quad \bar{P}_s = \int dx (\bar{T}_{s+1} + \bar{\Theta}_{s-1}) \]

where \( \{P_s\} \) is an infinite set of mutually commuting integrals of motion with integer spin \( s \).

Note \( H = P_1 + \bar{P}_1 \)
Assymptotic States

For an Integrable theory our Fock space is made up of assymptotic 'in' and 'out' states described by their rapidites $p_0 + p_1 = me^\theta$ and $p_0 - p_1 = me^{-\theta}$

\[
| A_{a_1}(\theta_1)A_{a_2}(\theta_2)\ldots A_{a_N}(\theta_N)\rangle = A_{a_1}(\theta_1)A_{a_2}(\theta_2)\ldots A_{a_N}(\theta_N) | 0\rangle
\]

where in states are ordered as $\theta_1 > \theta_2 > \ldots > \theta_N$ and out states as $\theta_1 < \theta_2 < \ldots < \theta_N$

Assymptotic states diagonalise the local Integrals of motion,

\[
[P_s, A_a(\theta)] = \gamma_a^{(s)} e^{s\theta} A_a(\theta) \quad [\bar{P}_s, A_a(\theta)] = \gamma_a^{(s)} e^{-s\theta} A_a(\theta)
\]

(note $\gamma_a^{(1)} = m_a$)
The S-Matrix

Scattering is purely elastic and the S-matrix is factorisable. Creation operators satisfy

\[ A_{a_1}(\theta_1)A_{a_2}(\theta_2) = S_{a_1a_2}^{b_1b_2}(\theta_1 - \theta_2)A_{b_2}(\theta_2)A_{a_1}(\theta_1) \]

We assume \( \mathbf{C}, \mathbf{P} \) and \( \mathbf{T} \) symmetries so \( S(\theta) \) satisfies

\[ S_{a_1a_2}^{b_1b_2}(\theta) = S_{\bar{a}_1\bar{a}_2}^{\bar{b}_1\bar{b}_2}(\theta) = S_{a_2a_1}^{b_2b_1}(\theta) = S_{b_2b_1}^{\bar{a}_2\bar{a}_1}(\theta) \]
Properties of the S-matrix

1. The Yang-Baxter equation

\[ S_{a_1 a_2}^{c_1 c_2}(\theta) S_{c_1 c_3}^{b_1 c_3}(\theta + \theta') S_{c_2 c_3}^{b_2 b_3}(\theta') = S_{a_2 a_3}^{c_2 c_3}(\theta') S_{a_1 c_3}^{c_1 b_3}(\theta + \theta') S_{c_1 c_2}^{b_1 b_2}(\theta') \]

2. Unitarity Condition

\[ S_{a_1 a_2}^{c_1 c_2}(\theta) S_{c_1 c_2}^{b_1 b_2}(-\theta) = \delta_{a_1}^{b_1} \delta_{a_2}^{b_2} \]

3. Analyticity and Crossing Symmetry

\[ S_{a_1 a_2}^{b_1 b_2}(\theta) = S_{a_2 b_1}^{b_2 \bar{a}_1}(i\pi - \theta) \]

and \( S_{a_1 a_2}^{b_1 b_2}(\theta) \) is meromorphic and real at \( \text{Im}\theta = 0 \).
Properties of the S-matrix

4. **Bootstrap Condition**

The only singularities of $S_{a_1a_2}^{b_1b_2}(\theta)$ in the physical strip are at $\text{Re}\theta = 0$ and interpreted as bound states. These are stable so must correspond to a particle $c$. Position of pole in the direct channel is $iu_{a_1a_2}^c$ then

$$m_{a_1}^2 + m_{a_2}^2 - m_c^2 = -2m_{a_1}m_{a_2}\cos u_{a_1a_2}^c$$

The pole term can be written as

$$S_{a_1a_2}^{b_1b_2}(\theta) \sim i \frac{f_{a_1a_2}^c f_{b_1b_2}^c}{\theta - iu_{a_1a_2}^c}$$

and the bootstrap equation is given by

$$f_{a_1a_2}^c S_{ca_3}^{bb_3}(\theta) = f_{c_1c_2}^b S_{a_1c_3}^{c_1b_3}(\theta + i\bar{u}_{a_1c}) S_{a_2a_3}^{c_2b_3}(\theta + i\bar{u}_{a_2\bar{c}})$$
Adding a Boundary

Now restrict $x \in (-\infty, 0]$ and add in a boundary field with a boundary action density. In the CFT picture this can be seen a perturbed conformal boundary condition. This gives the equation

$$T_{xy}|_{x=0} = (-i)(T - \bar{T})|_{x=0} = \frac{d}{dy} \theta(y)$$

This condition preserves translation in the $y$ direction and can also bee seen as a consequence of this.
An Integral of Motion

From the continuity equations

\[ P_1(C) = \int_C (T dz + \Theta d\bar{z}) \quad \bar{P}_1(C) = \int_C (\bar{T} d\bar{z} + \Theta dz) \]

are independent of \( C \) and thus \( P_1(C) = \bar{P}_1(C) = 0 \).

Splitting the contour as follows and evaluating

\[ P_1(C_{12}) + \bar{P}_1(C_{12}) = \theta(y_1) - \theta(y_2) \]

allows us to say that

\[ H_B(y) = \int_{-\infty}^0 (T + \bar{T} + 2\Theta) dx + \theta(y) \]

is \( y \)-independent and thus an integral of motion.
Choose boundary conditions such that

\[ [T_{s+1} + \bar{\Theta}_{s-1} - \bar{T}_{s+1} - \Theta_{s-1}]|_{x=0} = \frac{d}{dy} \theta_s(y) \]

Then by the argument above

\[ H^{(s)}_B(y) = \int_{-\infty}^{0} (T_{s+1} + \Theta_{s-1} + \bar{T}_{s+1} + \Theta_{s-1})dx + \theta_s(y) \]

is also an integral of motion.
Assymptotic States in the Boundary Theory

Conserved charges act on in(out) states as

\[ H^{(s)}_B |A_{a_1}(\theta_1)A_{a_2}(\theta_2)\ldots A_{a_N}(\theta_N)\rangle_{B, in(out)} = \]
\[ \left( \sum_{i=1}^{N} 2\gamma^{(s)}_{a_i} \cosh(s\theta_i) + h^{(s)} \right) |A_{a_1}(\theta_1)A_{a_2}(\theta_2)\ldots A_{a_N}(\theta_N)\rangle_{B, in(out)} \]

These conserved charges imply that the number of particles is conserved and that the out-going momenta are a permutation of minus the in-going momenta.

It is then possible to argue that the S-matrix is factoisable.
The Boundary Operator

The ground state in the boundary theory is formally written as

$$|0\rangle_B = B|0\rangle$$

The operator $$B : \mathcal{H} \mapsto \mathcal{H}_B$$ and satisfies

$$A_a(\theta)B = R^b_a(\theta)A_b(-\theta)B \quad [H_s, B] = h^{(s)}B$$

We can now express 'in' states in terms of 'out' states.
Correltion Functions

So far assumed boundary in space so Hilbert space is not the same as in bulk theory. Hamiltonian identified with $H_B = H_B^{(1)}$ and correlation functions computed as matrix elements

$$\langle O_1(x_1, y_1)...O_N(x_N, y_N) \rangle = \frac{B\langle 0|T_y(O_1(x_1, y_1)...O_N(x_N, y_N))|0\rangle_B}{B\langle 0|0\rangle_B}$$

In the Euclidean picture we can take out boundary to be a boundary in time or initial state $|B\rangle$. The Hilbert space is the same as the bulk theory and correlation functions calculated via

$$\langle O_1(x_1, y_1)...O_N(x_N, y_N) \rangle = \frac{\langle 0|T_x(O_1(x_1, y_1)...O_N(x_N, y_N))|B\rangle}{\langle 0|B\rangle}$$

Because of the integrals of motion in the boundary theory we must have

$$(P_s - \bar{P}_s)|B\rangle = 0$$
Boundary S-matrix Equations

1. The Boundary Yang-Baxter Equation

\[
R_{a_2}^{c_2}(\theta_2) S_{a_1 c_2}^{c_1 d_2}(\theta_1 + \theta_2) R_{c_1}^{d_1}(\theta_1) S_{d_2 d_1}^{b_2 b_1}(\theta_1 - \theta_2) = \\
S_{a_1 a_2}^{c_1 c_2}(\theta_1 - \theta_2) R_{c_1}^{d_1}(\theta_1) S_{d_2 d_1}^{d_1 b_1}(\theta_1 + \theta_2) R_{d_2}^{b_2}(\theta_2)
\]

2. Boundary Unitarity

\[
R_a^c(\theta) R_c^d(-\theta) = \delta_a^b
\]
3. **Crossing Symmetry** - Need to switch pictures. \((P_s - \bar{P}_s)\) acting on asymptotic states has eigen value

\[
\sum_{i=1}^{N} 2\gamma_{a_i}^{(s)} \sinh(s\theta_i)
\]

So particles can only appear in the boundary state in pairs with equal mass and opposite momentum.

\[
|B\rangle = \sum_{N=0}^{\infty} \int_{0<\theta_1<...<\theta_N} d\theta_1...d\theta_N K_{2N}^{a_N...a_1 b_1...b_N} (\theta_1, ..., \theta_N)
\]

\[
A_{a_N}(-\theta_N)...A_{a_1}(-\theta_1)A_{b_1}(\theta_1)...A_{b_N}(\theta_N)|0\rangle
\]

\[
= (1 + \frac{1}{2} \int_0^{\infty} d\theta K^{ab}(\theta) A_a(-\theta)A_b(\theta) + ...) |0\rangle
\]
With appropriately normalised $A_a(\theta)$ we have

$$K_{2N}^{a_1b_1...a_nb_n}(\theta_1,...,\theta_N) = R_{\bar{a}_1...\bar{a}_N}^{b_1...b_N}(\frac{i\pi}{2} - \theta_1,...,\frac{i\pi}{2} - \theta_1)$$

As 'in' and 'out' states are related via the S-matrix we have the 'boundary cross-unitarity condition'

$$K^{ab}(\theta) = S_{a'b'}^{ab}(2\theta)K^{a'b'}(-\theta)$$

All $K_{2N}$ can be expressed in terms of $K(\theta) = \frac{1}{2}K^{ab}(\theta)A_a(-\theta)A_b(\theta)$ so we can write

$$|B\rangle = \exp(\int_0^\infty d\theta K(\theta))|0\rangle$$
Boundary S-matrix Equations

4. **Boundary Bootstrap Conditions**—Boundary scattering of bound state particles.

\[ f_d^{ab} R_c^d(\theta) = f_c^{b_1 a_1} R_{a_1}^{a_2}(\theta + i\bar{u}_{ad}^b) S_{b_1 a_2}^{b_2 a}(2\theta + i\bar{u}_{ad}^b - i\bar{u}_{bd}^a) R_b^b(\theta - i\bar{u}_{bd}^a) \]

For 2 particles of equal mass expect a pole in \( R_{a}^{b} \) at \( \theta = \frac{i\pi}{2} - \frac{u_{ab}^c}{2} \) ie.

\[ K^{ab}(\theta) \approx \frac{i}{2} \frac{f_c^{ab} g^c}{\theta - iu_{ab}^c} \]

Non-zero \( g^c \) indicates that \( |B\rangle \) has a contribution from a zero momentum particle \( A_c \)

\[ |B\rangle = (1 + g^c A_c(0) + \frac{1}{2} \int_0^\infty d\theta K^{ab}(\theta) A_a(-\theta) A_b(\theta) + ...) |0\rangle \]
The Ising Model

\[
A = \int dx dy (\psi \bar{\partial} \psi - \bar{\psi} \partial \bar{\psi} + m \psi \bar{\psi})
\]

Take \( m > 0 \) as the low-temperature phase. Ground state is degenerate \(|0, \pm\rangle\) corresponding to the expectation values of the spin field \( \langle \sigma(x) \rangle_{\pm} = \pm \bar{\sigma} \).

Can expand \( \psi \) and \( \bar{\psi} \) as

\[
\psi(x, t) = \int d\theta [\omega e^{\theta/2} A(\theta) e^{imx \sinh(\theta) + imt \cosh(\theta)} + \bar{\omega} e^{\theta/2} A^\dagger(\theta) e^{-imx \sinh(\theta) - imt \cosh(\theta)}]
\]

\[
\bar{\psi}(x, t) = \int d\theta [\bar{\omega} e^{-\theta/2} A(\theta) e^{imx \sinh(\theta) + imt \cosh(\theta)} + \omega e^{-\theta/2} A^\dagger(\theta) e^{-imx \sinh(\theta) - imt \cosh(\theta)}]
\]
The S-matrix

Creation operators anti-commute:

\[ A^\dagger(\theta) A^\dagger(\theta') = -A^\dagger(\theta') A^\dagger(\theta) \]

So \( S = -1 \) and the boundary scattering amplitude satisfies

\[ A^\dagger(\theta) B = R(\theta) A^\dagger(-\theta) B \quad R(\theta) R(-\theta) = 1 \]

\[ K(\theta) = -K(-\theta) \quad K(\theta) = R\left(\frac{i\pi}{2} - \theta\right) \]
Fixed Boundary Conditions

Removes ground state degeneracy.

\[ (\psi + \bar{\psi})|_{x=0} = 0 \]

In terms of creation operators

\[ (\bar{\omega} e^{\theta/2} + \omega e^{-\theta/2}) A^\dagger(\theta) = -(\omega e^{\theta/2} + \bar{\omega} e^{-\theta/2}) A^\dagger(-\theta) \]

And so

\[ R_{\text{fixed}}(\theta) = i \tanh(\frac{i\pi}{4} - \frac{\theta}{2}) \]
The Fixed Boundary State

In this picture fields $\chi = \omega \psi$ and $\bar{\chi} = \overline{\omega} \overline{\psi}$ have the same decomposition and creation/annihilation operators as above and boundary conditions become

$$(\chi + i \bar{\chi})|\tau=0|B_{\text{fixed}}\rangle = 0$$

This gives

$$|B\rangle_{\text{fixed}} = \exp\left\{ \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}\theta K_{\text{fixed}}(\theta) A^\dagger(-\theta) A^\dagger(\theta) \right\} |0\rangle$$

and $K_{\text{fixed}}(\theta) = i \tanh(\frac{\theta}{2})$
Free Boundary Conditions

No restrictions on boundary spins.

\[(\psi - \bar{\psi})_{x=0} = 0\]

Which gives

\[R_{\text{free}}(\theta) = -i\coth\left(\frac{i\pi}{4} - \frac{\theta}{2}\right)\quad \text{and} \quad K_{\text{free}}(\theta) = -i\coth\left(\frac{\theta}{2}\right)\]

The boundary state is given by

\[|B_{\text{free}}\rangle = (1 + A^\dagger(0))\exp\left\{\frac{1}{2} \int_{-\infty}^{\infty} d\theta K_{\text{free}}(\theta)A^\dagger(-\theta)A^\dagger(\theta)\right\}|0\rangle\]
Boundary Magnetic Field

Boundary condition

\[ i \frac{d}{dy} (\psi - \bar{\psi})_x=0 = \frac{h^2}{2} (\psi + \bar{\psi})_x=0 \]

and

\[ R_h(\theta) = itanh\left(\frac{i\pi}{4} - \frac{\theta}{2}\right) \frac{\kappa - isinh(\theta)}{\kappa + isinh(\theta)} \]

where \( \kappa = 1 - \frac{h^2}{2m} \).
The Energy Operator

Use the Formula

$$\epsilon_0(t) = \sum_{n=0}^{\infty} \langle 0|\epsilon(x, t)|n \rangle \langle n|B \rangle$$

and the only non-zero matrix element is

$$\langle 0|\epsilon(x, t)|\beta_1\beta_2 \rangle = -2\pi m i \sinh(\frac{\beta_1 - \beta_2}{2})e^{-mt(cosh(\beta_1)+cosh(\beta_2))} + imx(sin\beta_1 + sin\beta_2)$$

From this we get

$$\epsilon_0(t, h) = -im \int_{0}^{\infty} d\beta \sinh(\beta) \hat{K}(\beta)e^{-2mtcosh(\beta)}$$