CFT dual of the AdS Dirichlet problem

Fluid/Gravity on cut-off surfaces

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We study the gravitational Dirichlet problem in AdS spacetimes with a view to understanding the boundary CFT interpretation.
Based on the paper: hep-th/1106.2577

Two threads...

Two major ideas feed into this work:

- Holography in the near horizon (dominated by Strominger's work)
- AdS-CFT and trivial flow

Two key papers

1. From Navier-Stokes To Einstein [1]
   - Asymptotically flat Einstein equation with cut-off
   - Navier-Stokes versus near horizon
   - Criticism: What is flat space holography?
2. From Petrov-Einstein to Navier-Stokes [2]
   - Petrov I condition = Navier-Stokes constraints

One approach

- The action:

$$S = -\frac{1}{2}\int d^{d+1}x\sqrt{-g}\frac{1}{q(r)}\left(\nabla\phi\right)^2$$

- The momentum:

$$\Pi(r, x^\mu) = -\frac{\sqrt{-g}}{q(r)}g^{rr}\partial_r\phi$$

- The membrane:

$$S_{bh} = \int_\Sigma d^d x\sqrt{-\gamma}\left(\frac{\Pi(r_0, x^\mu)}{\sqrt{-\gamma}}\right)\phi(r_0, x^\mu)$$

Hamiltonian flow

- The Hamiltonian equations of motion:

$$\begin{aligned} \partial_r \phi &= -\frac{q(r)}{\sqrt{-g}} g_{rr} \Pi \\ \partial_r \Pi &= \frac{\sqrt{-g}}{q(r)} g^{rr} g^{\mu\nu} k_\mu k_\nu \phi \end{aligned}$$

- The Green's function:

$$G = \lim_{r \to \infty} \frac{\Pi(r, x^\mu)}{\phi(r, x^\mu)}$$

- Dispersion:

$$\omega(k) = c_s k - i\Gamma k^2 + \ldots$$

Transport

- Decay:

$$\Gamma = \frac{2\eta}{3\left(\epsilon + P\right)}$$

- Hydrodynamics:

$$\begin{aligned} \partial_r \left(\omega\phi\right) &= 0 \\ \partial_r \Pi &= 0 \end{aligned}$$

- Criticism: Non-linear determination? Hard to extract transport coefficients.

The statement

"What is the boundary dual for solving the Dirichlet problem in an AdS spacetime?"

Or...

"What are the metric and stress tensor in the boundary when we impose that the bulk metric pulled back to a constant "r" slice have a given form?"

Problems and extensions

Problems we will fix:

- Boundary holography is well controlled (vis-a-vis Strominger's work)
- Will provide a complete specification of how fluid quantities flow from the horizon to the boundary (vis-a-vis)

We will also be able to extend the work of [3]

Spatially homogeneous probe scalar

- The equation of motion:

$$\frac{1}{r^{d+1}} d_r \left(r^{d+1} d_r \Phi_k \right) - \left(m^2 + k^2 \right) \Phi_k = 0$$

- Solution for $k = 0$:

$$\Phi = \frac{\langle O \rangle}{2\nu} r^{-\Delta} + \phi_0 r^{\Delta - d}$$

where Δ, m and ν are related by:

$$\begin{aligned} \Delta \left(\Delta - d \right) &= m^2 \\ \nu &= \Delta - \frac{d}{2} \end{aligned}$$

At $r = r_D$

- Dirichlet surface position $r = r_D$
- The current and expectation value

$$\begin{aligned} \langle \hat{O} \rangle &= \langle O \rangle \\ \hat{\phi}_0 &= \phi_0 + \frac{\langle O \rangle}{(2\nu)\, r_D^{2\nu}} \end{aligned}$$

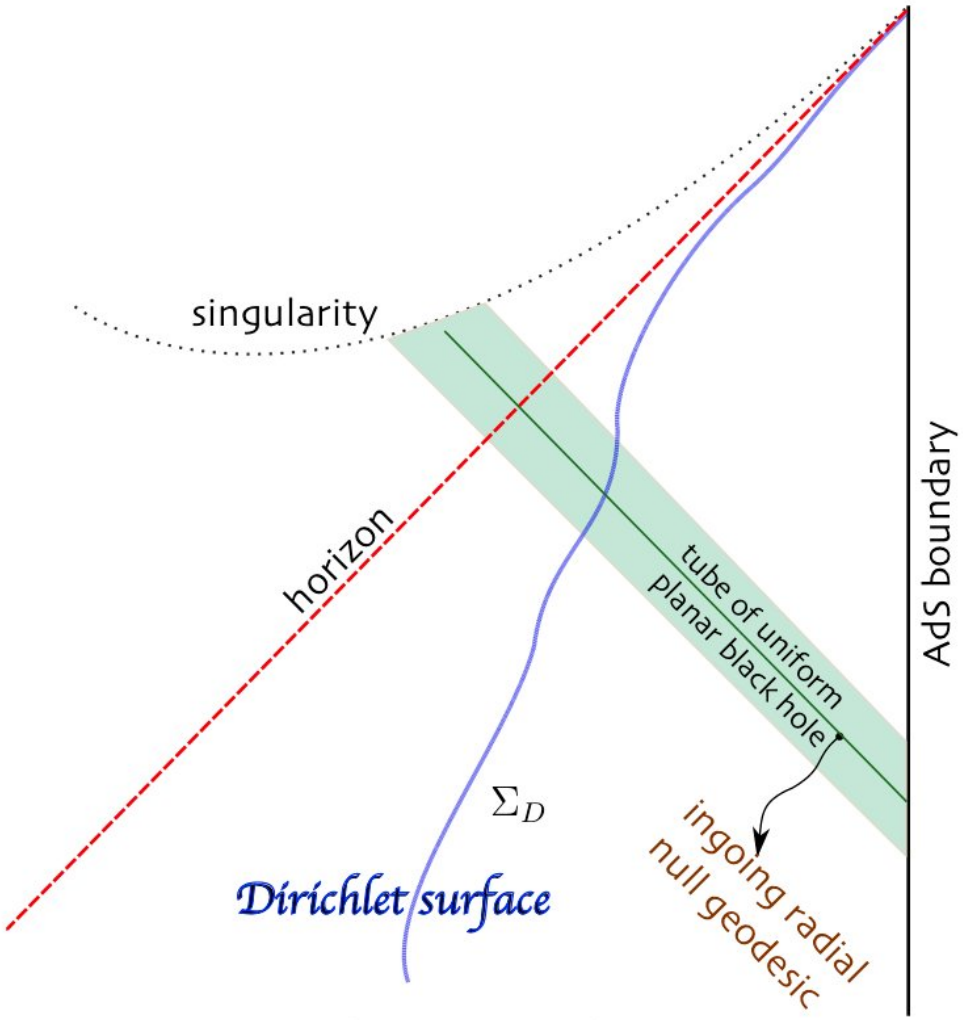
- The field:

$$\Phi = \hat{\phi}_0 r^{-\Delta} + \left(\hat{\phi}_0 - \frac{\langle O \rangle}{(2\nu)\, r_D^{2\nu}} \right) r^{\Delta - d}$$

- The boundary interpretation:

$$\begin{aligned} \mathcal{L}_{CFT} &= \hat{\phi}_0 \langle O \rangle \\ \delta \mathcal{L}_{CFT} &\propto \left(\hat{\phi}_0 - \frac{\langle O \rangle}{(2\nu)\, r_D^{2\nu}} \right) \langle O \rangle \end{aligned}$$

The picture



$$\hat{\mathfrak{X}} = \{\hat{g}_{\mu\nu}, \hat{u}^\mu, \hat{T}\} \quad \overset{\varphi_D}{\longmapsto} \quad \mathfrak{X} = \{g_{\mu\nu}, u^\mu, T\}$$

The metric and SEM tensor

- Use forced fluid dictionary where bulk metric is:

$$ds^2 = -2\tilde{u}_\mu(x)\left(dr + r\tilde{B}_\nu(r,x)dx^\nu\right) + r^2\tilde{G}_{\mu\nu}(r,x)dx^\mu dx^\nu$$

  The objects $\tilde{u}_\mu$, $\tilde{B}_\mu$ and $\tilde{G}_{\mu\nu}$ are Weyl covariant. Built from fluid velocity, u_μ, and boundary metric $g_{\mu\nu}$.

- SEM given by:

$$T_{\mu\nu} = -\frac{r_D^d}{8\pi G_{d+1}}\left(\hat{K}_{\mu\nu} - \hat{K}\hat{g}_{\mu\nu} + (d-1)\hat{g}_{\mu\nu} + \ldots\right)$$

  where $r_D^2\hat{K}_{\mu\nu}$ is extrinsic curvature of surface.

- Surface metric from pullback:

$$\hat{g}_{\mu\nu} = g_{\mu\nu} + \frac{u_\mu u_\nu}{(br_D)^d} + O(\partial)$$

  where $b \propto \frac{1}{T}$

Dictionary

The dictionary is un-enlightening to see but key points:

1. SEM tensor has same form as boundary
2. However non-zero trace given by:

$$T = -r_D \frac{d\hat{\epsilon}}{dr_D}$$

3. Thermodynamics:

$$\begin{aligned} \hat{\epsilon} &= \frac{d-1}{8\pi G_{d+1}} \frac{\hat{\alpha}}{\hat{\alpha}+1} \frac{1}{b^d} \\ \hat{\epsilon} + \hat{P} &= \frac{d}{16\pi G_{d+1}} \frac{\hat{\alpha}}{b^d} \end{aligned}$$

where:

$$\hat{\alpha} = \frac{1}{\sqrt{f\left(br_D\right)}}$$

4. Speed of sound:

$$c_s^2 = \hat{c}_s^2 \left(1 + \frac{d}{2}\left(\alpha^2 - 1\right)\right)$$

BMW limit

- To achieve non-relativistic limit in boundary we take scaling
- Write metric as:

$$g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu}$$

  to get forcing.
- Anistropic in space and time:

$$x \to \epsilon x \quad t \to \epsilon^2 t$$

  and:

$$u^\mu = \left(1, \epsilon v^i\right)$$

- Scale thermodynamics:

$$b = b_0 + \epsilon^2 \delta b$$

Output of scaling

- SEM conservation $\Rightarrow$ incompressible Navier-Stokes with forcing
- Replace above quantities with hats
- Now have non-relativistic version of fluid
- We match with [3] but go one order higher and have forcing

What's happening?

- Look at spatially projected relativistic conservation equations:

$$-\left(1+\frac{d}{2}\left(\hat{\alpha}^2-1\right)\right)\hat{P}^{\alpha}_{\mu}\frac{\hat{\nabla}_{\alpha}b}{b}+\hat{a}_{\mu}$$
$$-\frac{2b^d}{\hat{\alpha}d}\hat{P}_{\mu\alpha}\hat{\nabla}_{\beta}\left(\frac{1}{b^{d-1}}\hat{\sigma}^{\alpha\beta}\right) \quad = \quad 0$$

- $\hat{\alpha}=\frac{1}{\sqrt{f(br_D)}}$ blows up
- Vacuous dynamics
- Does this make sense?

A new scaling

- Attempt something similar to BMW except take:

$$T \to \hat{\alpha} T$$

  We get non-vacuous dynamics that are exactly incompressible Navier-Stokes
- But T seems to shrink to nothing
- Hydrodynamics is a really an expansion of the form:

$$\frac{\omega}{T}, \frac{k}{T} << 1$$

- Maybe we can switch to a near horizon parameter instead (vis-a-vis Strominger)
- What is boundary intepretation?
  1. Metric degenerates
  2. Newton-Cartan structure, $g^{\mu\nu}$ and t_μ instead
  3. Be careful with raising and lowering indices now

The take-home

"For hydrodynamic quantities the boundary is not a special place in AdS. By this I mean you can calculate quantities like energy density or viscosity in AdS with a finite cut-off "r", say by evolving equations of motion from the horizon, and then using our maps you can complete your spacetime with an AdS boundary."

Unsolved mysteries

- What is the precise nature of the membrane paradigm in the boundary CFT?
- How does the membrane translate to an effective $AdS_2 \times \mathbb{R}^{d-1}$ geometry at extremality? How is this effective IR-CFT realised in the boundary?
- Why does the speed of sound become one at some finite "r"?
- When we move beyond hydrodynamics how does the map between boundary and membrane change? We should expect non-trivial "re-normalisation" flow...

References

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* Temporary entry *.

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