M-Theory and Matrix Models

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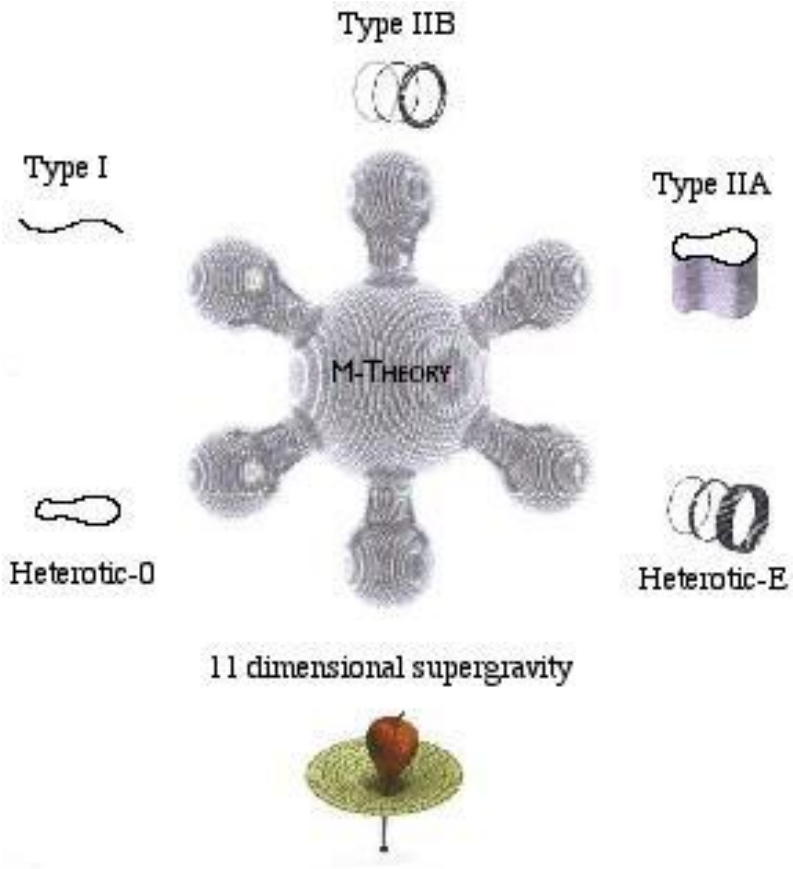
Superstrings

There are 5 different superstring theories which describe open and closed strings



D-Branes

- There are dualities between the 5 different theories, for example T-duality describes how we can relate IIA to IIB theories.
- Open strings have end points that live on D-branes
- D-p-branes are (p+1)-dimensional objects that move through D dimensional spacetime, they can have open/closed bc's

D-Branes

- A bosonic brane action would typically look like the following

$$S_{DBI} = T_p \int \mathrm{d}^{p+1}\sigma \sqrt{-\det P(G + 2\pi\alpha' F)}$$

  These branes have a tension which is some power of g_s and l_s
- A p-brane couples to a p+1 form potential, you can add these potentials to the D-brane action

$$S = S_{DBI} + \int A_{p+1}$$

The Discovery of M-Theory

- M-theory was discovered as a strongly coupled limit to IIA theory
- This comes from a Kaluza Klein argument where

$$M_{11}^2 = -p_m p^m = 0, \quad p_{11} = N/R_{11},$$

  where M_{11} is the mass of the graviton in 11D SUGRA/M-Theory
- The D0-brane has mass $(l_s g_s)^{-1}$ and KK modes have N units of $U(1)$ charge

The Discovery of M-Theory

- Matching up the two masses gives $R_{11} = l_s g_s, \quad g_s \to \infty$ gives us extra large dimension
- Eleven dimensional SUGRA has a 3-form potential
- M-theory has the same potential, yielding an electrically coupled M2 and a magnetically coupled M5 brane.
- Worldvolume theories for these branes would be 3 and 6 dimensional CFTs

AdS/CFT... Because no strings talk is complete without it

- We can look at the AdS/CFT correspondence here too!
- The correspondence tells us that a stack of M2-branes has bulk background $AdS_4 \times S^7$, this was known from the 11-dimensional supergravity result of the M2.
- Bagger and Lambert discovered the theory of multiple M2-branes in 2007
- In 11 dimensions $\mathcal{N} = 8$ is the maximal supersymmetry we could have.

Whats new in M-Theory

- BL proposed that the $\mathcal{N} = 8$, $\mathrm{SO}(8)_R$ theory of multiple M2 branes is a *Chern-Simons gauge theory* of level k.
- The revolutionary idea was that the scalars X^I took values in a 3-algebra $[\cdot,\cdot,\cdot] : V \times V \times V \to V$.

$$\mathcal{L} = -\frac{1}{2}D^\mu X^I D_\mu X^I + \frac{i}{2}\bar{\Psi}\Gamma^\mu D_\mu \Psi + \frac{i}{4}\bar{\Psi}\Gamma_{IJ}[X^I, X^J, \Psi]$$
$$- V(X) + \frac{k}{2}\varepsilon^{\mu\nu\lambda}(\tilde{A}_\mu \partial_\nu \tilde{A}_\lambda + \frac{2}{3}\tilde{A}_\mu \tilde{A}_\nu \tilde{A}_\lambda)$$

where

$$V(X) = \frac{1}{12}\mathrm{Tr}([X^I, X^J, X^K], [X^I, X^J, X^K])$$

Whats new in M-Theory

- Using the notation $\tilde{A}_\mu{}^c{}_d = A_{\mu ab} f^{abc}{}_d$, we can write the susy transforms:

$$\begin{aligned} \delta X^I &= i\bar{\epsilon}\Gamma^I\Psi \\ \delta\Psi &= D_\mu X^I \Gamma^\mu \Gamma^I \epsilon - \frac{1}{6}[X^I, X^J, X^K]\Gamma^{IJK}\epsilon \\ \delta\tilde{A}_\mu &= i\bar{\epsilon}\Gamma_\mu\Gamma_I X^I \Psi, \end{aligned}$$

- M2-branes are $1/2$ BPS and so we choose $\Gamma^{012}\epsilon = \epsilon$
- Let's discuss a bit more about Lie 3-algebras

Whats new in M-Theory

- A Lie 3-algebra is a vector space with a basis $T^a,\ a = 1, \ldots, N$ endowed with a trilinear product

$$[T^a, T^b, T^c] = f^{abc}{}_d\, T^d,$$

- We can derive a generalised Jacobi Id called the fundamental identity by using a derivation on this product.
- The $\mathcal{N} = 8$ theory required $f^{abc}{}_d = f^{[abc]}{}_d$, only solution prop to ϵ^{abcd}. So BL theory has an $SO(4) \cong SU(2) \times SU(2)$ gauge symmetry.
- Reformulate as a bi-fundamental gauge theory.

Whats new in M-Theory

- BL theory only describes **two** coincident branes! Also the Chern-Simons level is restricted to $k = 1, 2$
- ABJM wrote down a *bifundamental* SCCS matter theory which has *only* $\mathcal{N} = 6$ susy, but this described N coincident branes for any integer k.
- The moduli space of the ABJM theory is $\mathbb{R}^8/\mathbb{Z}_k, k \in \mathbb{Z}$ for a U(N) $\times$ U(N) gauge theory.
- ABJM theory does not need to use 3 algebras, the potential of the theory is calculated from a superpotential in the usual manner.

Whats new in M-Theory

- It is possible to enhance the $\mathcal{N} = 6$ ABJM theory to get the full $\mathcal{N} = 8$ theory by the use of monopole operators.
- The M2-brane picture is pretty well understood now.
- On to the M5-brane...

Whats new in M-Theory

- Much less is known about the M5-brane.
- Gravity side has $AdS_7 \times S^4$.
- Field theory is given by an $\mathcal{N} = (2,0)$ supermultiplet $(B_{\mu\nu}, X^I, \Psi)$, $I = 6, ..., 10$.
- Difficult to write a gauge kinetic term for the field strength $H_3 = dB_2$.
- $\int_{\Sigma_6} H_3 \wedge H_3 = 0$
- Possible to write down a non-covariant action for a single M5-brane which is gauge fixed, (PST 1997).

Introduction

- Expect Riemannian geometry to break down at Plank scales.
- Spacetime becomes quantised and coordinates become quantum operators.
- Open strings ending on D-branes with a B-field turned on give a noncommutative geometry

$$[X^\mu, X^\nu] = i\theta^{\mu\nu}.$$

Introduction

- M2-branes end on M5-branes. Has been shown that M5-brane geometry in the presence of 3-form C-field is

$$[X^\mu, X^\nu, X^\lambda] = i\theta^{\mu\nu\lambda}.$$

- Lie 3-bracket given by fundamental identity:

$$[[f, g, h], k, l] = [[f, k, l], g, h] + [f, [g, k, l], h] + [f, g, [h, k, l]]$$

- Constraint is too strong and we do not understand the representation theory.

Introduction

- We propose the 'Quantum Nambu Bracket':

$$[f, g, h] := fgh + ghf + hfg - fhg - gfh - hgf$$

- We will show that the quantum geometry $[X^\mu, X^\nu, X^\lambda] = i\theta^{\mu\nu\lambda}$ is given by the QNB.
- Refer to the above geometry as the 'Quantum Nambu Geometry'.

A II B Background

- Consider background $AdS_5 \times S^5$.
- Background has a dilaton, axion, C_2, C_4.
- Turn on 3-form Flux for a D1-brane in the AdS_5 sector only.
- It has been shown that consistency requires the spacetime to take the form
$$\mathbb{R}^3 \times AdS_2 \times S^5.$$
- Note for later that the RR 2-form potential is given by
$$C_2 = f\epsilon_{ijk}X^i dX^j dX^k.$$

D1 MM in a Large Flux

- N parallel D1-branes action given by DBI + CS action
- Myers proposed a generalisation of the CS term given by

$$S_{CS} = \mu_1 \int \mathrm{Tr} P(e^{i\lambda \mathrm{i}_\Phi \mathrm{i}_\Phi} \sum_n C_n) e^{\lambda F}.$$

- Here $\mu_1 = 1/(g_s 2\pi\alpha')$, $\lambda = 2\pi\alpha'$ and $X^I = 2\pi\alpha'\Phi^I$. Our background has $B = 0$.

D1 MM in a Large Flux

- Expanding S_{CS} gives

$$\begin{aligned} S_{CS} &= \mu_1 \int \mathrm{Tr}[\lambda F \chi + C_2 + i\lambda^2 F \mathrm{i}_\Phi \mathrm{i}_\Phi C_2 \\ &\qquad + i\lambda \mathrm{i}_\Phi \mathrm{i}_\Phi C_4 - \frac{\lambda^3}{2} F \mathrm{i}_\Phi^4 C_4] \\ &:= S_\chi + S_{C_2} + S_{C_4} \end{aligned}$$

- We will see only S_{C_2} contributes in the double scaling limit of a large field strength.

D1 MM in a Large Flux

- Substituing our expression for C_2 gives

$$\begin{aligned} S_{C_2}/\mu_1 &= f\int d^2\sigma \mathrm{Tr}(\frac{1}{2}\epsilon_{ijk}X^i D_\alpha X^j D_\beta X^k \epsilon^{\alpha\beta}) \\ &\quad + f\int d^2\sigma \mathrm{Tr}(iFX^i X^j X^k \epsilon_{ijk}) \end{aligned}$$

- Here $\alpha = 0, 1,\ i = 2, 3, 4,\ i' = 5, 6, 7, 8, 9.$

D1 MM in a Large Flux

- By including a YM term (from DBI), can find EOM

contribution of:	S_{C_2}	$\mathrm{i}_\Phi^2 C_4$	$\mathrm{i}_\Phi^4 C_4$
EOM of X^i:	$O(\frac{f}{\alpha'})$	$O(\frac{1}{f\alpha'^2})$	0
EOM of $X^{i'}$:	0	$O(\frac{1}{f^4\alpha'^2})$	$O(\frac{1}{f^4\alpha'^2})$

- The YM term is given by

$$S_{YM}/\mu_1 = \alpha'^2 \int \sqrt{-\det G_{\alpha\beta}}\, F_{\alpha\beta} F_{\alpha'\beta'} G^{\alpha\alpha'} G^{\beta\beta'}$$

- So for large f and taking $\epsilon \to 0$, we have a double scaling limit

$$\begin{aligned} \alpha' &\sim \epsilon, \\ f &\sim \epsilon^{-a}, \quad a > 0. \end{aligned}$$

D1 MM in a Large Flux

- Choice $1/2 < a < 2$ gives us

$$\lim_{\epsilon \to 0} S_{D1} = S_{C_2}.$$

- So the double scaling limit gives us the low energy action for N D1-branes in a large F_3 background.

Equations of Motion and Quantum Nambu Geometry

$$\varepsilon^{\alpha\beta}\varepsilon_{ijk}[X^j, D_\beta X^k X^i] + \varepsilon^{\alpha\beta}\varepsilon_{ijk}[D_\beta, X^i X^j X^k] = 0,$$
$$\frac{3}{2}\varepsilon_{ijk} D_\alpha X^j D_\beta X^k \varepsilon^{\alpha\beta} + \varepsilon_{ijk}[F; X^j, X^k]' = 0,$$

- $[A; B, C]' := [B, C]A + A[B, C] + BAC - CAB$
- $\mathrm{Tr}[A, B, C]D = \mathrm{Tr}[D; B, C]'A$, in analogy to the relation $\mathrm{Tr}D[A, B] = \mathrm{Tr}[D, A]B$.

Equations of Motion and Quantum Nambu Geometry

Solutions:

- $D_\alpha X^i = 0$,
- $F = 0$,
- $[X^i, X^j] = i\theta\epsilon^{ij}$ is allowed but there is another more interesting solution,
- $[X^i, X^j, X^k] = i\theta\epsilon^{ijk}$

A Representation with Z_3 Symmetry

- Can construct a rep with 'raising/lowering' operators.
- Look at a different rep, want to consider reps on a 3-dim real space.

$$[X^1, X^2, X^3] = i\theta$$

- Introduce a unitary operator,

$$\begin{aligned} U|\omega\rangle &= \quad |\rho^2\omega\rangle, \quad \rho^3 = 1 \in \mathbb{C} \\ U^\dagger|\omega\rangle &= \quad |\rho\omega\rangle, \end{aligned}$$

and assuming

$$X^1|\omega\rangle = (\omega + a)|\omega + 1\rangle,$$

one obtains

$$\begin{aligned} U^\dagger X^1 U|\omega\rangle &= \quad (\rho^2\omega + a)|\omega + \rho\rangle, \\ U^{\dagger 2} X^1 U^2|\omega\rangle &= \quad (\rho\omega + a)|\omega + \rho^2\rangle \end{aligned}$$

A Representation with Z_3 Symmetry

- If the fields X^1, X^2 and X^3 are unitarily related to each other by

$$\begin{aligned} X^2 &= U^\dagger X^1 U, \\ X^3 &= U^\dagger X^2 U, \end{aligned}$$

then

$$\begin{aligned} X^1|\omega\rangle &= (\omega + a)|\omega + 1\rangle, \\ X^2|\omega\rangle &= \rho^2(\omega + a\rho)|\omega + \rho\rangle, \\ X^3|\omega\rangle &= \rho(\omega + a\rho^2)|\omega + \rho^2\rangle \end{aligned}$$

and we see that

$$[X^1, X^2, X^3]|\omega\rangle = 3(a^2 - a)(\rho - \rho^2)|\omega\rangle$$

IIB Matrix Model

- Expand around the QNG and examine emerging gauge theories
- Denote the covariant derivative $iD^\alpha = i\partial^\alpha + A^\alpha$, $\alpha = 0, 1$ as

$$iD^\alpha = X^\alpha$$

- We can show that S_{D1} can be written as

$$S_{D1} = \frac{f}{g_s l_s^2} \int d^2\sigma \, \mathrm{Tr} X^a X^b X^c X^d X^e \epsilon_{abcde},$$

- Large N reduction gives

$$S_{IIB} = \frac{f}{g_s l_s^2} \mathrm{Tr} X^a X^b X^c X^d X^e \epsilon_{abcde}, \quad a, b, c, d, e = 0, 1, 2, 3, 4.$$

M-Theory Matrix Model

- IIB background invariant under KV $\partial_i, i = 2, 3, 4$.
- Compactify x^2 on circle of radius R_2 and T-dualize

$$S^1 \times \mathbb{R}^2 \times AdS_2 \times S^5$$

- D1-brane $\rightarrow$ D2-brane under this T-duality
- In IIA theory with D2-brane action

$$S_{D2} = \frac{f}{g_s l_s} \int d^3\sigma X^a X^b X^c X^d X^e \epsilon_{abcde}$$

M-Theory Matrix Model

- In II A theory, we have D0, D2, D4, D6, D8-branes.
- M-theory on a circle in the infinite momentum frame gives us only states with positive D0-charge
- Under the same double scaling, we are left with

$$S_{D0} = \frac{1}{g_s l_s} \int P(C^{(1)}) = \frac{f}{g_s l_s} \int dt \epsilon_{ij} X^i D_t X^j, \quad i, j = 3, 4$$

- BFSS model describes M-theory in infinite momentum frame using QM - D0-branes

M-Theory Matrix Model

- D2-action describes D0-branes too
- Dimensionally reduce D2-action
- In the large flux curved background, M-theory action is

$$S_M = \frac{if}{g_s l_s} \int dt \, \mathrm{Tr} D_t X^b X^c X^d X^e \epsilon_{bcde}, \quad b, c, d, e = 1, 2, 3, 4$$

where $X^0 = -iD_t$

II A Matrix Model

- Write constants in terms of M-theory radius R_{11}, compactify on x^2, perform 11-2 flip
- The flip changes the M-theory direction on the torus. All this means is

$$R_2 = g_s l_s, \qquad R_{11} = N$$

with lightcone momentum $p_+ = 1$

- Obtain the II A matrix model

$$S_{IIA} = \frac{f}{N} \int d^2\sigma \, \mathrm{Tr} X^a X^b X^c X^d X^e \epsilon_{abcde},$$

where $a, b, c, d, e = 0, 1, 2, 3, 4$

Multiple D4-branes

- Consider a fluctuation around the QNG

$$X^i = x^i \mathbf{1}_{K\times K} + A^i(\sigma, x^j)$$

- The IIA action becomes

$$S_5 = \frac{f}{N}\int_{\Sigma_5} \mathrm{tr} X^a X^b X^c X^d X^e \epsilon_{abcde}$$

where $\int_{\Sigma_5} = \int d^2\sigma \int_x$ and $\int_x$ is an integral on QNG

Multiple D4-branes

- Argue this describes K parallel D4-branes
- Introduce 3-form H

$$\begin{aligned} H^{abc} &= -i[X^a, X^b, X^c], \\ H^{de5} &= -i[X^d, X^e], \qquad a, b, c, d, e = 0, 1, 2, 3, 4, \end{aligned}$$

where

$$H^{*\mu\nu\lambda} := \frac{1}{6\sqrt{-g}} \epsilon^{\mu\nu\lambda\rho\alpha\beta} H_{\rho\alpha\beta}$$

is the Hodge dual of $H_{\rho\alpha\beta}$.

Multiple D4-branes

- Claim

$$S_5 = \int_{\Sigma_5} \mathrm{tr} H^{abc} H^{de5} \, \epsilon_{abcde}$$

  describes the D4's. To see the connection, we examine an abelian M5-brane theory and dimensionally reduce
- PST wrote the action for a single M5-brane in 6d by coupling to a scalar.

Multiple D4-branes

- Gauge fixing gives us the following non-covariant action

$$S_{PST} = -\frac{1}{4} \times \int d^6\sigma \left(\frac{1}{6} \epsilon_{abcde} H^{abc} H^{de5} + H^{*ab5} H^{*}_{ab5} \sqrt{-g} \right)$$

- After dim reduction on x^5, 1st term is precisely our action!
- 2nd term comes from a YM term, this is neglected in double scaling limit

Multiple M5-branes

- Recent proposal that M5-branes on S^1 is equivalent to D4-branes with instantons.
- There is a precise mapping between instanton states and KK modes
- Seems natural to promote $X^5 = \mathbf{1}$ to a general field

$$H^{\mu\nu\lambda} = -i[X^\mu, X^\nu, X^\lambda]$$

- Different from field strength of 2-form formulation, here in terms of a 1-form
- Two formulations are related non-trivially

Multiple M5-branes

- Propose action

$$S_{M5} = -\frac{1}{4} \int_{\Sigma_6} \mathrm{tr}\Big(\frac{1}{6}\epsilon_{abcde}H^{abc}H^{de5} + \big(c_2 H^{abc}H_{abc} + c_3 H^{ab5}H_{ab5}\big)\sqrt{-g}\Big)$$

where $\Sigma_6 = \Sigma_5 \times S^1$

- H must satisfy a Bianchi Id. Most natural one is

$$[X^{[\mu}, H^{\nu\lambda\rho]}] = 0$$

- Requiring self-duality solves EOM

$$H_{ab5} = -\frac{1}{6}\epsilon_{abcde}H^{cde}$$

Multiple M5-branes

- The solution for coefficients are given by 1-parameter family

$$c_2 = -\frac{1}{18}(c_1/2), \quad c_3 = \frac{5}{6}(c_1/2)$$

- Moreover, the self-duality condition on H implies that the QNG is self-dual!

$$[x^\mu, x^\nu, x^\lambda] = i\theta^{\mu\nu\lambda}\mathbf{1}$$

Multiple M5-branes

- The final action for K parallel M5-branes (or K non-abelian 3-form) is

$$S_{M5,\theta} = -\frac{1}{4}\int_x \mathrm{tr}\Big(\frac{1}{6}\epsilon_{abcde}H^{abc}H^{de5} + \Big(\alpha\,\frac{-1}{3}H^{abc}H_{abc} + (1-\alpha)H^{ab5}H_{ab5}\Big)\sqrt{-g}\Big)$$

  where $\int_x$ is determined by the rep of the QNG.
- Only $\alpha = 1/6$ gives us the correct Bianchi Id and self-duality condition.
- The last two terms combine to give the non-abelian generalisation of the gauge fixed PST action!

Conclusion

Results:

- New kind of geometry in string theory
- D1's in a large flux $\rightarrow$ non-abelian 3-form on M5's.
- Identify $C^{\mu\nu\lambda} = \theta^{\mu\nu\lambda}$

Future:

- More representation theory
- QFT on a QNG
- Supersymmetry