# Outline

- What is an instanton?
- How do we find instantons?
- The geometry on the space of instantons.
- Dynamics of slow moving instantons.

# Instantons on D4-branes

James Allen

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# Where are we?

## String Theory

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### String Theory

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### Solitons

- A long time ago: Instantons
- Also a while ago: Non-perturbative physics
- Recently: Instantons hold part of the key to understanding M5-branes?

# Why is this interesting?

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In the reduction, the Kaluza-Klein modes should be identified with instantons. However, we have to be careful with the singularities that arise if we want to quantise the theory.

The maths is cool and the calculations are fun.

# What is an instanton?

We're sitting on a D4-brane so our theory is 4+1 Yang-Mills,

$$S = \int \mathrm{d}^5 x \ \mathrm{Tr}\left(\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \tfrac{1}{2} D_\mu \phi D^\mu \phi\right).$$

Everything is classical, so we only need the equations of motion,

$$\begin{aligned} D^\mu F_{\mu\nu} + [D_\mu \phi, \phi] &= 0, \\ D^2 \phi &= 0. \end{aligned}$$

# What is an instanton? (cont.)

Every solution has an energy associated with it,

$$E = \int \mathrm{d}^4x \ \mathrm{Tr}\left( \tfrac{1}{2} F_{i0} F_{i0} + \tfrac{1}{2} D_0 \phi D_0 \phi + \tfrac{1}{4} F_{ij} F_{ij} + \tfrac{1}{2} D_i \phi D_i \phi \right)$$

Kinetic and Potential

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Kinetic and Potential

The minimum energy solution is the vacuum, but that's pretty boring. There are other minimum energy solutions fixed by the *topological* nature of the theory.

# What is an instanton? (cont.)

Recall the kinetic and potential energy density terms for the gauge field, $F_{i0}F_{i0}$ and $F_{ij}F_{ij}$. For finite energy, these must go to zero at spatial infinity.

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$$A_\mu(x)\big|_{|x|=\infty} = g(x)\,\partial_\mu g^{-1}(x),$$

where

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Maps like this have an integer topological degree, $k$, which is how many times they 'wind' round $S^3$. This splits our theory into sectors with different topological degrees at infinity.

Each topological sector will have a minimum energy solution.
These are *instantons*!

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$$
\begin{aligned}
E = {} & 2\pi^2|k| + |Q_E| \\
& + \int \mathrm{d}^4x \ \mathrm{Tr}\left(\tfrac{1}{2}(F_{i0} - D_i\phi)^2 + (F_{ij} - \tfrac{1}{2}\varepsilon_{ijkl}F_{kl})^2\right).
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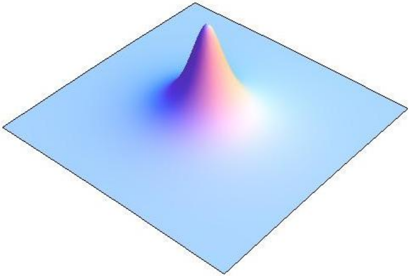
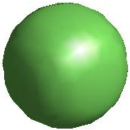
So instantons satisfy,

$$F_{ij} = \tfrac{1}{2}\varepsilon_{ijkl}F_{kl} \\ D_i\phi = F_{i0}.$$

# What does an instanton look like?

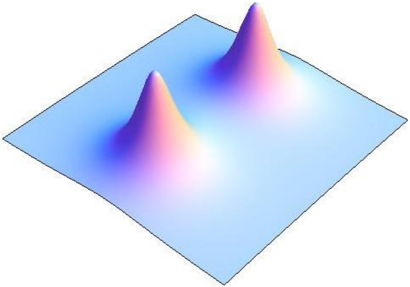


The scalar field profile of an instanton. Unfortunately I can only show two dimensions of the $4$-dimensional space this is living on.

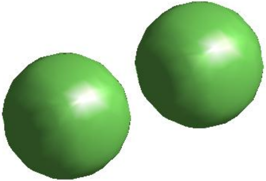


An isosurface of the instanton charge density. I do slightly better here by showing three dimensions, but only one value.

# What about two instantons?

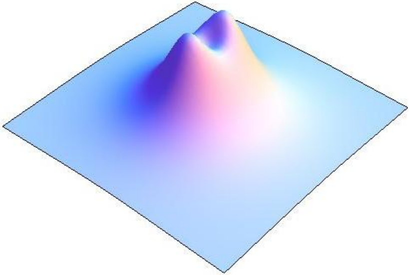


The scalar field of two instantons. It looks just like two copies of a single instanton!

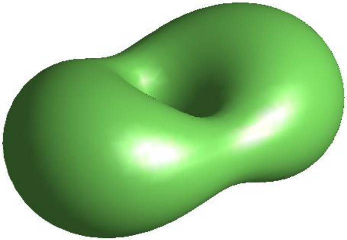


Instanton charge density.

# Two instantons close together



The scalar field of two close instantons. They aren't as obviously separate now.



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# How can I ~~make pretty plots~~ find instanton solutions?

With the *ADHM construction*.

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With the *ADHM construction*.

Instead of solving a horrible differential equation for the components of $A_\mu$, we can instead solve some algebraic constraints.

I don't know how to motivate this procedure other than just showing you it, so stick with me until we get to the end of this section and all will become clear.

# The ADHM Construction

We start with a quaternionic matrix,

$$\Delta(x) = \begin{pmatrix} v_1 & v_2 \\ \rho + \tau & \sigma \\ \sigma & \rho - \tau \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} x$$

The parameters encode the following information about the instantons:

| | |
|---|---|
| $\lvert v_1\rvert$ and $\lvert v_2\rvert$ | The sizes of the instantons |
| $\hat{v}_1$ and $\hat{v}_2$ | The gauge rotation of each instanton |
| $\rho$ | The centre of mass |
| $\tau$ | The separation of the instantons. |

# The ADHM Construction (cont.)

To find $A_i$, we first find an orthonormal basis of null vectors of $\Delta$, and group them together as $U$,

$$\Delta^\dagger U = 0, \quad \text{and} \quad U^\dagger U = 1_2.$$

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The gauge field is then given by

$$A_i = iU^\dagger \partial_i U,$$

and

$$F_{ij} = -U^\dagger b(e_i f \bar{e}_j - e_j f \bar{e}_i) b^\dagger U$$

is self-dual if $f = (\Delta^\dagger \Delta)^{-1}$ is real.

# The moduli space

The ADHM construction gives us *all* instantons of a given charge! This means that the space of instantons is finite dimensional, and has a coordinate system given by the parameters in the ADHM data. We call this space the *moduli space*.

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Caveat. The ADHM construction only gives us a unique instanton up to a local gauge transformation. The moduli space is really all the instanton solutions quotiented by local gauge transformations so that each point is physically different. We can pick a representative instanton at each point.

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Caveat to the caveat. The global gauge transformations turn out to be quite important so we will still include those in the moduli space.

# Moduli Space Geometry

A tangent vector is like a an infinitesimal variation which leaves you in the space. If we vary $A_i \to A_i + \delta A_i$ then $\delta A_i$ must satisfy the linearised self-dual equation:

$$D_{[i}\delta A_{j]} = \varepsilon_{ijkl} D_k \delta A_l.$$

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Any variation $\delta A_i$ satisfying this can be though of as a tangent vector to the moduli space. There is a natural inner product which gives us our metric:

$$g(\delta A_i, \delta' A_i) = \int \mathrm{d}^4x \,\mathrm{Tr}\left(\delta A_i \delta' A_i\right).$$

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For a consistent metric, we have to project orthogonal to gauge transformations:

$$D_i \delta A_i = 0.$$

# Metric from ADHM data

The ADHM parameters, $z^r = v_1^i, v_2^i, \tau^i$, provided a natural coordinate system on the moduli space and the tangent vectors in this coordinate system are

$$\delta_r A_i = \partial_r A_i - D_i \epsilon_r,$$

where $\epsilon_r$ is chosen so that $D_i \delta_r A_i = 0$.

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Once again the ADHM construction turns the problem of calculating the metric in an algebraic problem,

$$\delta_r A_i = \partial_r A_i = U^\dagger \delta_r \Delta f \bar{e}_i b^\dagger U - U^\dagger b e_i f \delta_r \Delta^\dagger U.$$

# James rolls up his sleeves

The metric on the two instanton moduli space is

$$ds^2 = dv_1^2 + dv_2^2 + d\tau^2 + d\sigma^2 - \frac{k^2}{N_A},$$

where

$$\begin{aligned} k &= \bar{v}_1 dv_2 - \bar{v}_2 dv_1 + 2(\bar{\tau} d\sigma - \bar{\sigma} d\tau) \\ N_A &= |v_1|^2 + |v_2|^2 + 4\left(|\tau|^2 + |\sigma|^2\right). \end{aligned}$$

For $\Delta^\dagger \Delta$ to be real we must have

$$\sigma = \frac{\tau}{4|\tau|^2}(\bar{v}_2 v_2 - \bar{v}_1 v_2).$$

# James *really* rolls up his sleeves

$$
\begin{aligned}
\mathrm{d}s^2 = \Big[ & \mathrm{d}v_1^2 + \mathrm{d}v_2^2 + \mathrm{d}\tau^2 \\
& + \frac{1}{4|\tau|^2}\Big( |v_1|^2\,\mathrm{d}v_2^2 + |v_2|^2\,\mathrm{d}v_1^2 + 2(v_1\cdot\mathrm{d}v_1)(v_2\cdot\mathrm{d}v_2) \\
& \qquad - (v_1\cdot\mathrm{d}v_2)^2 - (v_2\cdot\mathrm{d}v_1)^2 - 2(v_1\cdot v_2)(\mathrm{d}v_1\cdot\mathrm{d}v_2) \\
& \qquad + 2\varepsilon_{ijkl}v_1^i v_2^j\,\mathrm{d}v_1^k\,\mathrm{d}v_2^l\Big) \\
& + \frac{1}{4|\tau|^4}\Big(|v_1|^2|v_2|^2 - (v_1\cdot v_2)^2\Big)\,\mathrm{d}\tau^2 \\
& - \frac{1}{2|\tau|^4}\Big(|v_1|^2(v_2\cdot\mathrm{d}v_2) + |v_2|^2(v_1\cdot\mathrm{d}v_1) \\
& \qquad - (v_1\cdot v_2)(v_1\cdot\mathrm{d}v_2) - (v_1\cdot v_2)(v_2\cdot\mathrm{d}v_1)\Big)\tau\cdot\mathrm{d}\tau \\
& + \frac{1}{8|\tau|^4}\Big(\varepsilon_{ijkl}\Lambda_i\,\mathrm{d}\Lambda_j\,\tau_k\,\mathrm{d}\tau_l \\
& \qquad + (\Lambda\cdot\mathrm{d}\tau)(\tau\cdot\mathrm{d}\Lambda) - (\Lambda\cdot\tau)(\mathrm{d}\Lambda\cdot\mathrm{d}\tau)\Big)\tau\cdot\mathrm{d}\tau \\
& - \frac{1}{N_A}\Bigg(v_1\cdot\mathrm{d}v_2 - v_2\cdot\mathrm{d}v_1 \\
& \qquad - \frac{2}{|\tau|^2}\big(\varepsilon_{mnpq}v_2^m v_1^n \tau^p\,\mathrm{d}\tau^q + (v_2\cdot\tau)(v_1\cdot\mathrm{d}\tau) - (v_1\cdot\tau)(v_2\cdot\mathrm{d}\tau)\big)\Bigg)^2\Bigg]
\end{aligned}
$$

# So what do we do with it?

We can approximate the motion of slow moving instantons by motion on the moduli space. Motion close to the bottom of a valley floor can be approximated by motion entirely along the valley floor, so long as we only ever climb a small way up the sides. The effective action for slow moving instantons is

$$S = \frac{1}{2}\int dt\, g_{rs}\dot{z}^r\dot{z}^s - |v|^2 g_{rs}K^r K^s + \mathcal{O}(\dot{z}^2 v^2)$$

So now we can ask what happens when we push two instantons towards each other and see how they interact.

# Interesting propertied

- Singularities on the moduli space.
  The moduli space has a conversed angular momentum,

  $$L = |v_1|^2\dot{\theta}_1 + |v_2|^2\dot{\theta}_2 + \text{interaction terms.}$$

  For a single instanton this prevents $|v|$ from passing through zero and keeps away from the singularity. Does this happen for interacting instantons?

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- Right-angled scattering.
- Geodesic submanifolds of the moduli space.

# Conclusions

It turns out that we can readily find configurations of two instantons in which one shrinks to zero size. For one instanton a conserved angular momentum prevents this from happening, but for two intereacting instantons this protection is gone.

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It turns out that we can readily find configurations of two instantons in which one shrinks to zero size. For one instanton a conserved angular momentum prevents this from happening, but for two intereacting instantons this protection is gone.

By examining the symmetries of the ADHM data we see a clear mechanism for right-angled scattering. This appears in the dynamical scattering process frrequently as we would expect from other soliton systems.

# Future work

The same calculation can in principle be performed for SU(N) and this might give us some insight into the expected $N^3$ degrees of freedom of the M5-branes.

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The same calculation can in principle be performed for SU(N) and this might give us some insight into the expected $N^3$ degrees of freedom of the M5-branes.

We can also look at larger charge instantons. Finding symmetric solutions would be particularly pleasing, as this is motivated by the relation between instantons and skyrmions.

Thank you