# Periodic Monopoles 

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## SDYM

Consider the Yang-Mills action

$$
S=\int_{\mathbb{R}^{4}} \operatorname{Tr}(F \wedge * F)
$$

where $F$ is the 2-form field strength of the gauge potential $A$. We can find finite energy stationary points by using the identity

$$
\operatorname{Tr}(F \wedge * F)=\operatorname{Tr}(F \wedge F)+\frac{1}{2} \operatorname{Tr}((F-* F) \wedge *(F-* F))
$$

resulting in the Bogomolny energy bound

$$
S \geq \int_{\mathbb{R}^{4}} \operatorname{Tr}(F \wedge F)
$$

which is saturated when

$$
* F=F
$$

In 4 dimensions the self-duality equations describe instantons.

## Dimensional Reduction of SDYM

Many other integrable systems can be obtained from SDYM. In components,

$$
F=* F \quad \Longrightarrow \quad F_{\mu \nu}=\frac{1}{2} \epsilon_{\mu \nu \alpha \beta} F_{\alpha \beta}
$$

where

$$
F_{\mu \nu}=\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu}+\left[A_{\mu}, A_{\nu}\right]
$$

- Imposing independence on $x^{4}$ gives the Bogomolny equations for a monopole in $\mathbb{R}^{3}$

$$
F=* D \Phi \quad \Longrightarrow \quad F_{i j}=\epsilon_{i j k}\left(\partial_{k} \Phi+\left[A_{k}, \Phi\right]\right)
$$

where we identify $\Phi=A_{4}$ and $D_{i} \Phi=\partial_{i} \Phi+\left[A_{i}, \Phi\right]$. Hence $\Phi$ is a Lie algebra-valued adjoint scalar.

- What happens if we do this again?


## Dimensional Reduction of SDYM

- Independece of two dimensions gives the Hitchin equations,

$$
D_{\bar{s}} \Phi=0 \quad \text { and } \quad F_{s \bar{s}}=\frac{\mathrm{i}}{4}\left[\Phi^{\dagger}, \Phi\right]
$$

where $s, \bar{s}$ are complex coodinates, $s=x^{1}+\mathrm{i} x^{2}$.

- Once more gives the Nahm equations,

$$
\partial_{s} A_{i}=\frac{1}{2} \epsilon_{i j k}\left[A_{j}, A_{k}\right]
$$

- and finally,

$$
\left[A_{1}, A_{2}\right]=\left[A_{3}, A_{4}\right] \quad\left[A_{1}, A_{3}\right]=\left[A_{4}, A_{2}\right] \quad\left[A_{1}, A_{4}\right]=\left[A_{3}, A_{2}\right]
$$

which are relevant to the ADHM construction.
As we shall see, most of these dimensional reductions will be useful to us!

## Magnetic Monopoles

Consider the Yang-Mills-Higgs action

$$
S=\int_{\mathbb{R}^{3}} \operatorname{Tr}(F \wedge * F)+\operatorname{Tr}(D \Phi \wedge * D \Phi)+V(\Phi)
$$

We again find stationary points by a Bogomolny argument, resulting in

$$
S \geq 2 \int_{\mathbb{R}^{3}} \operatorname{Tr}(F \wedge D \Phi) \quad F=* D \Phi
$$

Magnetic charge can be seen to arise by defining the Abelian field strength $f=\operatorname{Tr}(F \Phi)$ and the definition $B=* F$. Gauss' theorem then gives the monopoole charge as a surface integral. Note the fields are asymptotically Abelian.

## Higgs Field

Derrick scaling means pure YM only has finite energy non-trivial solutions in $d=4$. In $d=2$ or 3 we need a Higgs field $\Phi$ to stabilise the solution.

The asymptotic behaviour of $\Phi$ determines the topology and symmetry breaking of the monopole. For $S U(2)$ the gauge symmetry is broken to $U(1)$. So there is a map

$$
\Phi_{\infty}: S_{\infty}^{2} \mapsto S^{2} \cong S U(2) / U(1)
$$

of degree

$$
\pi_{2}\left(S^{2}\right)=\mathbb{Z}
$$

The monopole number is also the first Chern number of the gauge field,

$$
N=\frac{\mathrm{i}}{2 \pi} \int_{S_{\infty}^{2}} f
$$

## Nahm Transform - forward

The Nahm transform is a powerful tool to solve the Bogomolny equations. It can be thought of as a generalised Fourier transform and is an adaptation of the ADHM construction for instantons. [Corrigan \& Goddard '84]

- First, consider normalised solutions of the twisted Dirac-like operator

$$
\left(\sigma_{j} \otimes D_{j}-\mathbf{1}_{2} \otimes(\mathrm{i} \Phi(\mathbf{x})+s)\right) v(\mathbf{x}, s)=0
$$

- The three Nahm matrices are given by

$$
T_{i}(s)=\int_{\mathbb{R}^{3}} x^{i} v^{\dagger}(\mathbf{x}, s) v(\mathbf{x}, s) \mathrm{d}^{3} x
$$

- They satisfy the Nahm equations in the interval $s \in[-1,1]$

$$
\partial_{s} T_{i}=\frac{\mathrm{i}}{2} \epsilon_{i j k}\left[T_{j}, T_{k}\right] .
$$

## Nahm Transform - inverse

- For the inverse, we consider solutions of

$$
\left(\mathbf{1}_{2 k} \otimes \frac{\mathrm{~d}}{\mathrm{~d} s}-\left(T_{j}+x_{j} \mathbf{1}_{k}\right) \otimes \sigma_{j}\right) \Psi(\mathbf{x}, s)=0
$$

- From which we obtain monopole fields gauge equivalent to those we started with:

$$
\Phi(\mathbf{x})=\mathrm{i} \int_{-1}^{1} s \Psi^{\dagger}(\mathbf{x}, s) \Psi(\mathbf{x}, s) \mathrm{d} s \quad A_{i}(\mathbf{x})=\int_{-1}^{1} \Psi^{\dagger}(\mathbf{x}, s) \partial_{i} \Psi(\mathbf{x}, s) \mathrm{d} s
$$

Solving the Nahm equations is usually simpler than the Bogomolny equations - but the inverse transform is hard and must often be approximated or performed numerically.

## Nahm Transform - Properties

- It takes you between two systems satisfying the SDYM equations.
- Rank of gauge group and topological charge are permuted (so charge 1 solitons give Abelian Nahm equations!)
- The monopole is located at $x_{i}=-\operatorname{Tr}\left(T_{i}\right)$.
- The relevant 4-manifolds can be related by the example of the 4-torus which is self-reciprocal under Nahm transform, with the four radii inverted [Braam \& van Baal '89]:

$$
T^{4} \rightarrow \hat{T}^{4} \quad \mathbb{R}^{3} \rightarrow \mathbb{R} \quad \mathbb{R}^{4} \rightarrow \bullet \quad \cdots
$$

- Nahm data has singularities where the Dirac operators are not invertible. This depends on the asymptotic behaviour of the monopole fields.


## String Theory Picture

A stack of $k$ D1-branes held between two D3-branes gives rise to a charge $k S U(2)$ monopole on the worldvolume of each D3-brane [Diaconescu '97]. Dirac monopoles correspond to semi-infinite D1-branes extending outside the D3s.

Imposing D-terms vanish gives the Nahm equations satisfied along the D1-brane. This confirms the interpretation of the Nahm matrices as describing the position of the D1-branes on the D3.

Similar setup for the periodic monopole, consisting of D3-branes between NS5-branes, gives Hitchin equations on a cylinder. [Cherkis \& Kapustin '03]

|  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| D3 | $\times$ | $\times$ | $\times$ | $\times$ |  |  |  |  |  |  |
| D1 | $\times$ |  |  |  | $x$ |  |  |  |  |  |

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| NS5 | $\times$ | $\times$ | $\times$ | $\times$ | $x$ | $x$ |  |  |  |  |
| D3 | $\times$ | $x$ | $x$ |  |  |  | $x$ |  |  |  |

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| D5 | $\times$ | $x$ | $x$ | $x$ | $x$ | $x$ |  |  |  |  |
| D3 | $\times$ | $x$ | $x$ |  |  |  | $x$ |  |  |  |

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| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| D4 | $\times$ | $x$ | $x$ |  | $x$ | $x$ |  |  |  |  |
| D4 | $\times$ | $x$ | $x$ | $x$ |  |  | $x$ |  |  |  |

## Periodic Solutions

Consider arranging instantons or monopoles in a lattice. This is equivalent to compactifying on a torus. The Nahm transform allows us to consider the self-duality equations on the reciprocal lattice (N.B. the Nahm correspondence has only been proved for some of these!) [Jardim '04]
instanton
periodic instanton
doubly periodic instanton
triply periodic instanton monopole periodic monopole
'physical space' 'Nahm space'

$$
\begin{array}{cc}
\mathbb{R}^{4} & \bullet \\
\mathbb{R}^{3} \times S^{1} & \hat{S}^{1} \\
\mathbb{R}^{2} \times T^{2} & \hat{T}^{2} \\
\mathbb{R} \times T^{3} & \hat{T}^{3} \\
\mathbb{R}^{3} & \mathbb{R} \\
\mathbb{R}^{2} \times S^{1} & \mathbb{R} \times \hat{S}^{1}
\end{array}
$$

There are three self-reciprocal possibilities: $T^{4}, \mathbb{R} \times T^{2}$ (doubly periodic monopole), $\mathbb{R}^{2}$ (vortices?)

## Chains of Instantons

A chain of instantons in 4 Euclidean dimensions can be interpreted as an instanton at finite temperature, or caloron.

One can construct the caloron fields by simply taking a suitable superposition of the fields obtained from the Ansatz of Corrigan, Fairlie and 't Hooft. [Harrington \& Shepard '78]

The caloron has some interesting limits:

- The BPS monopole is obtained as the caloron size $\rightarrow \infty$.
- We recover the single instanton if size $\ll$ period.
- Far from the chain the fields become 3-dimensional.

Topologically, calorons have an instanton charge and various monopole charges.

## Chains of Instantons

Recall the asymptotic behaviour of a monopole determines its symmetry breaking and topology. For periodic systems we look at the holonomy in the periodic direction,

$$
V\left(x_{i}\right)=\mathrm{e}^{\int_{x_{0}} A_{\mu} \mathrm{d} x^{\mu}}
$$

The Harrington-Shepard caloron has trivial holonomy. There are more interesting examples which tend to the $\mathrm{H}-\mathrm{S}$ caloron in the 'massless monopole' (no symmetry breaking) limit. [Lee \& Lu '98]

In this case an $S U(m)$ caloron splits into $m$ constituent monopoles. The interpretation is that a caloron is a monopole with gauge group an extended Lie group. [Garland \& Murray '88]

## Chains of Monopoles

Monopoles have been studied in a similar context. But there are problems:

- A similar procedure used to obtain the Harrington-Shepard caloron gives a divergent infinite sum.
- The energy diverges logarithmically with distance (the fields fall as $1 / r^{2}$ ), so it is not clear how to implement a Bogomolny argument.

Nevertheless, we can still look for solutions to the Bogomolny equations on $\mathbb{R}^{2} \times S^{1}$ with boundary conditions motivated by a chain of Dirac monopoles, so looks Abelian from a distance. [Cherkis \& Kapustin '03]

The Nahm data satisfies Hitchin's equations on a cylinder $\mathbb{R} \times \hat{S}^{1}$ :

$$
\hat{D}_{\bar{s}} \hat{\phi}=0 \quad \hat{F}_{s \bar{s}}=\frac{i}{4}\left[\hat{\phi}^{\dagger}, \hat{\Phi}\right] .
$$

## Chains of Monopoles

Take coordinates $z \in[0,2 \pi)$ in the periodic direction and $\zeta, \zeta^{*}$ in the transverse directions. The asymptotic holonomy of $A-i \Phi \mathrm{~d} z$ is

$$
V(\zeta) \sim \operatorname{diag}\left(\zeta^{\ell} \mathrm{e}^{\mathfrak{v}}+\mathcal{O}\left(\zeta^{\ell-1}\right), \zeta^{-\ell} \mathrm{e}^{-\mathfrak{v}}+\mathcal{O}\left(\zeta^{-\ell-1}\right)\right)
$$

where $\ell$ and $\mathfrak{v}$ are given by the asymptotics and $\ell$ is the monopole charge.
The coordinates on monopole space and those on Nahm space are related by the spectral curve polynomial

$$
\operatorname{det}\left(\mathrm{e}^{2 \pi s}-V(\zeta)\right)=0
$$

The asymptotic holonomy diverges for $\ell \neq 0$, so the Hitchin cylinder $\mathbb{R} \times \hat{S}^{1}$ is smooth.

## Chains of Monopoles

For $S U(2)$ charge 1 we have $\ell=1$, so there are no singularities in the Nahm data. The Hitchin equations are Abelian and solve to [Ward '05]

$$
\hat{A}_{r}=0 \quad \hat{A}_{t}=\text { i.const } \quad \hat{\Phi}=C \cosh (2 \pi s)
$$

For the inverse Nahm transform we must find two independent solutions of

$$
\left(\begin{array}{cc}
2 \partial_{\bar{s}}-z & \hat{\Phi}-\zeta \\
\hat{\Phi}^{*}-\zeta^{*} & 2 \partial_{s}+z
\end{array}\right) \psi=0
$$

As long as we stay away from the points $\zeta= \pm C, \psi$ has Gaussian solutions localised around $\pm s_{0}= \pm\left(\cosh ^{-1}(\zeta / C)\right) /(2 \pi)$. The monopole fields are then approximately

$$
A_{\zeta}=\frac{\zeta}{4 \sqrt{\zeta^{2}-C^{2}}} \sigma_{3} \quad A_{\zeta^{*}}=-\left(A_{\zeta}\right)^{*} \quad-\left(A_{z}+\mathrm{i} \Phi\right)=\frac{1}{2 \pi} \cosh ^{-1}\left(\frac{\zeta}{C}\right) \sigma_{3}
$$

## Chains of Monopoles

We can plot the energy density $\mathcal{E}=\nabla^{2}|\Phi|^{2}$ in the plane $z=$ const:
$\mathcal{E} \propto\left(r^{4}-2 r^{2} C^{2} \cos (2 \theta)+C^{4}\right)^{-1 / 2}$


There are two energy peaks! Can they be interpreted as constituents?

## Chains of Monopoles

The 2-d fields can be derived from a potential $\phi$ satisfying the Poisson equation with sources at $\pm C$ :
$A_{r}=-\frac{1}{r} \partial_{\theta} \phi \quad A_{\theta}=r \partial_{r} \phi \quad \Rightarrow \quad F_{r \theta}=\nabla^{2} \phi=\delta(\zeta-C)+\delta(\zeta+C)$.
Equivalently, we can compute the first Chern number,

$$
c_{1}=\frac{1}{2 \pi} \int_{\partial \mathbb{R}^{2}=S^{1}} A_{\theta} \mathrm{d} \theta
$$

and we find that each peak contributes a factor of 1 .
The magnetic field, $F_{r \theta}$, is concentrated near the points $\zeta= \pm C$. As $C$ is increased, our approximation improves: it works closer to the flux tubes, and $z$ dependence quickly becomes weaker.

## Chains of Monopoles

I've been looking at several extensions:

- limits of $C$ (can we recover the single monopole?),
- higher charge,
- larger gauge groups,
- Nahm transform for vortices,
- Dirac singularities,
- comparison with calorons and doubly periodic instantons,
- dynamics on Nahm space.


## Chains of Monopoles

I've been looking at several extensions:

- limits of $C$,
- higher charge (dynamics and moduli space),
- larger gauge groups,
- Nahm transform for vortices,
- Dirac singularities,
- comparison with calorons and doubly periodic instantons,
- dynamics on Nahm space.


## Chains of Monopoles

I've been looking at several extensions:

- limits of $C$,
- higher charge,
- larger gauge groups (also have constituents),
- Nahm transform for vortices,
- Dirac singularities,
- comparison with calorons and doubly periodic instantons,
- dynamics on Nahm space.


## Chains of Monopoles

I've been looking at several extensions:

- limits of $C$,
- higher charge,
- larger gauge groups,
- Nahm transform for vortices (expect to be self-reciprocal),
- Dirac singularities,
- comparison with calorons and doubly periodic instantons,
- dynamics on Nahm space.


## Chains of Monopoles

I've been looking at several extensions:

- limits of $C$,
- higher charge,
- larger gauge groups,
- Nahm transform for vortices,
- addition of magnetically charged Dirac singularities,
- comparison with calorons and doubly periodic instantons,
- dynamics on Nahm space.


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The Nahm space is now more interesting than a line, so it would be interesting to see whether the solutions of the Hitchin equations behave like soltions. Varying monopole parameters we can get $90^{\circ}$ scattering on the cylinder. But how about the topology?

## Chains of Monopoles

A charge 2 periodic monopole has been constructed. [Harland \& Ward '08] Here the Nahm data is $U(2)$ and we can take the Hitchin Higgs field to be off-diagonal, with (as before)

$$
\operatorname{det} \hat{\Phi}=C \cosh (2 \pi s)
$$

Note $\cosh (2 \pi s)$ has two zeros, at $s= \pm \mathrm{i} / 4$. We find two different solutions, according to whether or not the zeros are in the same component of $\hat{\phi}$. The solutions have different dynamics, with orthogonal scattering directions.

Do this by studying gauge transformations of the inverse Nahm transform,

$$
\left(\begin{array}{cc}
2 D_{\bar{s}}-z & \hat{\Phi}-\zeta \\
\hat{\Phi}^{*}-\zeta^{*} & 2 D_{s}+z
\end{array}\right) \psi=0
$$

Open questions: do the fields again become independent of $z$ in the large size/period limit? How do the constituents interact?

## Chains of Monopoles

As monopoles are obtained from the large scale limit of a periodic instanton, we expect periodic monopoles to be some limit of the instanton on $\mathbb{R}^{2} \times T^{2}$.

A doubly periodic instanton has been constructed [Ford \& Pawlowski '01] which splits into two periodic monopole constituents when one period is much greater than the other.

The Nahm data consists of Hitchin equations on $\hat{T}^{2}$ with two singularities and is solved in terms of the doubly periodic Jacobi $\vartheta$-function. How does this fit in with the smooth Nahm data of the periodic monopole?

Recall we can relate the coordinates in monopole space, $\zeta$, to those on the Nahm cylinder, $s$, by

$$
\mathrm{e}^{4 \pi s}-\zeta \mathrm{e}^{2 \pi s}+1=0
$$

## Chains of Monopoles

The map $\zeta \rightarrow 1 / \zeta^{*}$ inverts the monopole fields in the circle $r=1$ and the resulting spectral curve can be interpreted as that of a BPS monopole with two opposite Dirac singularities at the origin.

The boundary conditions now give two singularities on $\mathbb{R} \times \hat{S}^{1}$ which behave like the regions $s \rightarrow \pm \infty$ on the original cylinder.

Our periodic monopole thus seems to arise from the doubly periodic instanton where one of the constituents has been sent to $\infty$.

In the string theory picture, a second compactification requires the introduction of new D1-branes, which become the Dirac monopoles, or extra roots of the gauge group, in the periodic monopole limit.

The second Chern class of the doubly periodic instanton becomes the magnetic charge of the monopole.

## Summary

- Nahm transform takes us from solutions of dimensionally reduced SDYM on one manifold to solutions on a reciprocal manifold with respect to the 4-torus.
- The transformed system is easier to solve exactly, but the inverse transform is hard.
- Periodic arrays of monopoles and instantons usually show the appearance of lower dimensional constituents.
- Qualitative behaviour is similar for all periodic systems.


## END

